Reminders

- Course project progress report 1 due 2/27 → 2/29
- HW3 due 2/29
- PRA4 due 3/14
- Come to OH for course project discussion!

Artificial Intelligence Methods for Social Good

Lecture 12:

Basics of Reinforcement Learning

17-537 (9-unit) and 17-737 (12-unit)

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Outline

- Recap: Markov Decision Process (MDP)
- POMDP and Ranger Patrol Problem
- Reinforcement Learning (RL)
  - Q-Learning
  - Policy Gradient

Learning Objectives

- Understand the concept of
  - Reinforcement Learning
- Describe the following algorithms
  - Q-Learning
  - ϵ-Greedy
  - Policy Gradient
- For the adaptive ranger patrol problem, understand
  - Motivation
  - Task being solved
  - MDP Model for the problem

Recap: Markov Decision Process (MDP)

- Special case of sequential decision-making problems
- MDP = (S, A, T, R, γ)
  - S: set of states, $s_t \in S$ (where can agent be?)
  - A: set of actions, $a_t \in A$ (what can agent do?)
  - T: transition function $T(s_t, a_t, s_{t+1}) = \mathbb{P}(s_{t+1} | s_t, a_t)$ (what happens next?)

Next state only depends on the current state, not previous states! (Markovian)

  - R: reward function $r_t = R(s_t)$ or $R(s_t, a_t)$ or $R(s_t, a_t, s_{t+1})$ (what do I gain?)
  - $\gamma \in [0, 1]$ (discount factor)

Recap: Markov Decision Process (MDP)

- MDP = (S, A, T, R, γ)
- (Deterministic) Policy $\pi: S \rightarrow A$
  - Maps state to action, defines a *plan*
- Given a policy π, we can sample history $h = \{s_0, a_0, s_1, a_1, \ldots\}$
- Goal: find π to maximize utility
  - Expected
  - $\pi^* = \text{argmax}_\pi \mathbb{E}_{h\sim\pi}\left[\sum_t \gamma^t R\big(s_t, \pi(s_t)\big)\right]$
- Myopic strategy does not work: a_t affects future states

Recap: Value Function

- The value function of a given policy π describes the expected accumulated reward with discount starting from a state

$$V^{\pi}(s) = \mathbb{E}[\sum_{t=0}^{\infty} \gamma^t R\big(s_t, \pi(s_t)\big)]$$

Where $s_0 = s$

Sometimes called state value function or V-value function

Recap: Q-Value Function

- Similar to state value function $V^{\pi}(s)$, but defined on state-action pair
- $Q^{\pi}(s,a)$: expected total reward from state s onward if taking action a in state s, and follow policy π afterward

That is $Q^{\pi}(s,a) = R(s) + \gamma \sum_{s'} P(s'|s,a) V^{\pi}(s')$

Obviously $V^{\pi}(s) = Q^{\pi}(s,\pi(s))$

So $Q^{\pi}(s,a) = R(s) + \gamma \sum_{s'} P(s'|s,a) Q^{\pi}(s',\pi(s'))$

Recap: Bellman Equation

- $V^*(s), Q^*(s,a)$ satisfy the following Bellman Equation

$$V^*(s) = \max_{a \in A} \left[R(s,a) + \gamma \sum_{s'} \mathbb{P}(s'|s,a) V^*(s') \right]$$

$$Q^*(s,a) = R(s) + \gamma \sum_{s'} P(s'|s,a) \max_{a'} Q^*(s',a')$$

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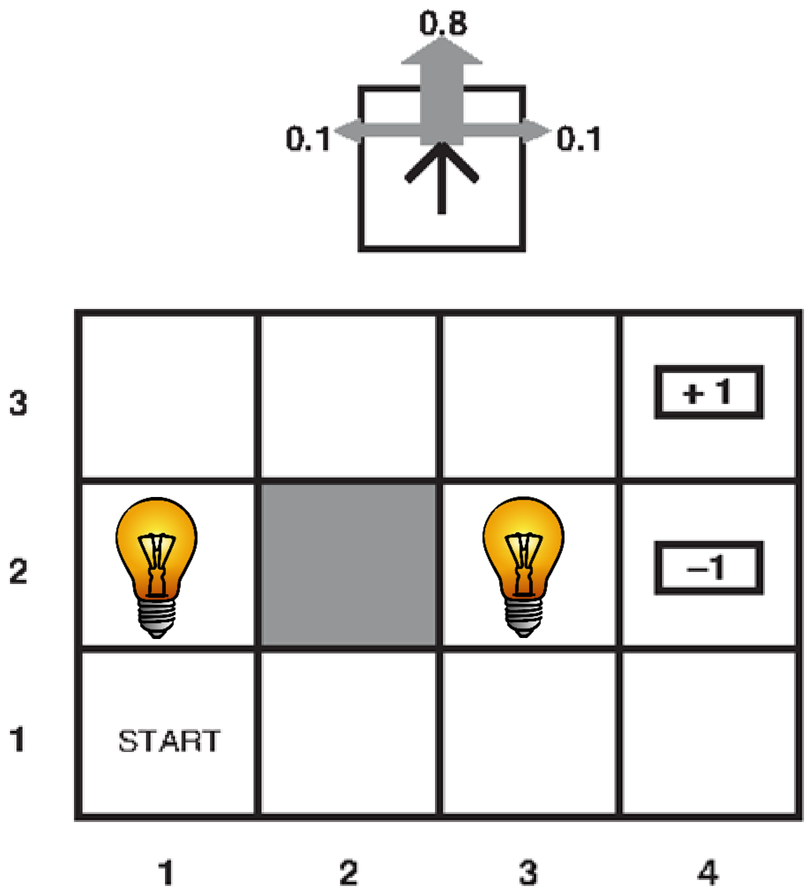
Partially Observable MDP (POMDP)

- POMDP = $(S, A, T, R, \Omega, O, \gamma)$
  - Ω: A set of possible observations
  - O: Observation probabilities: $o_t \sim O(o|s_{t+1})$
  - Agent is at state s_t (unknown to agent), takes action a_t, gets reward r_t, ends up at state s_{t+1}, and observes o_t

Example

MDP = $(S, A, T, R, \Omega, O, \gamma)$

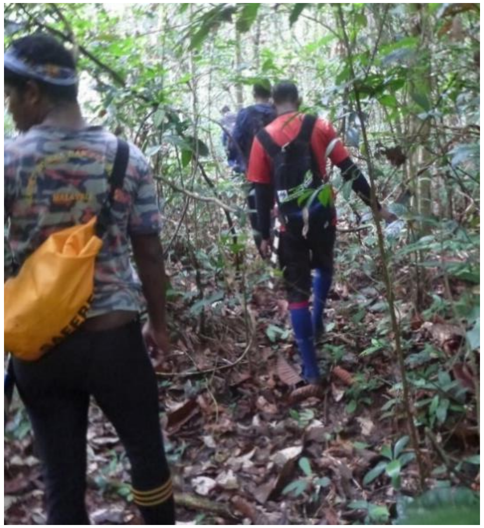


Ω ={Dark, Light}

Actions = {Up, Down, Left, Right}

Example: Ranger Patrol with Real-Time Information

Example: Ranger Patrol with Real-Time Information



Animal Footprint

Tiger Sign

Lighter

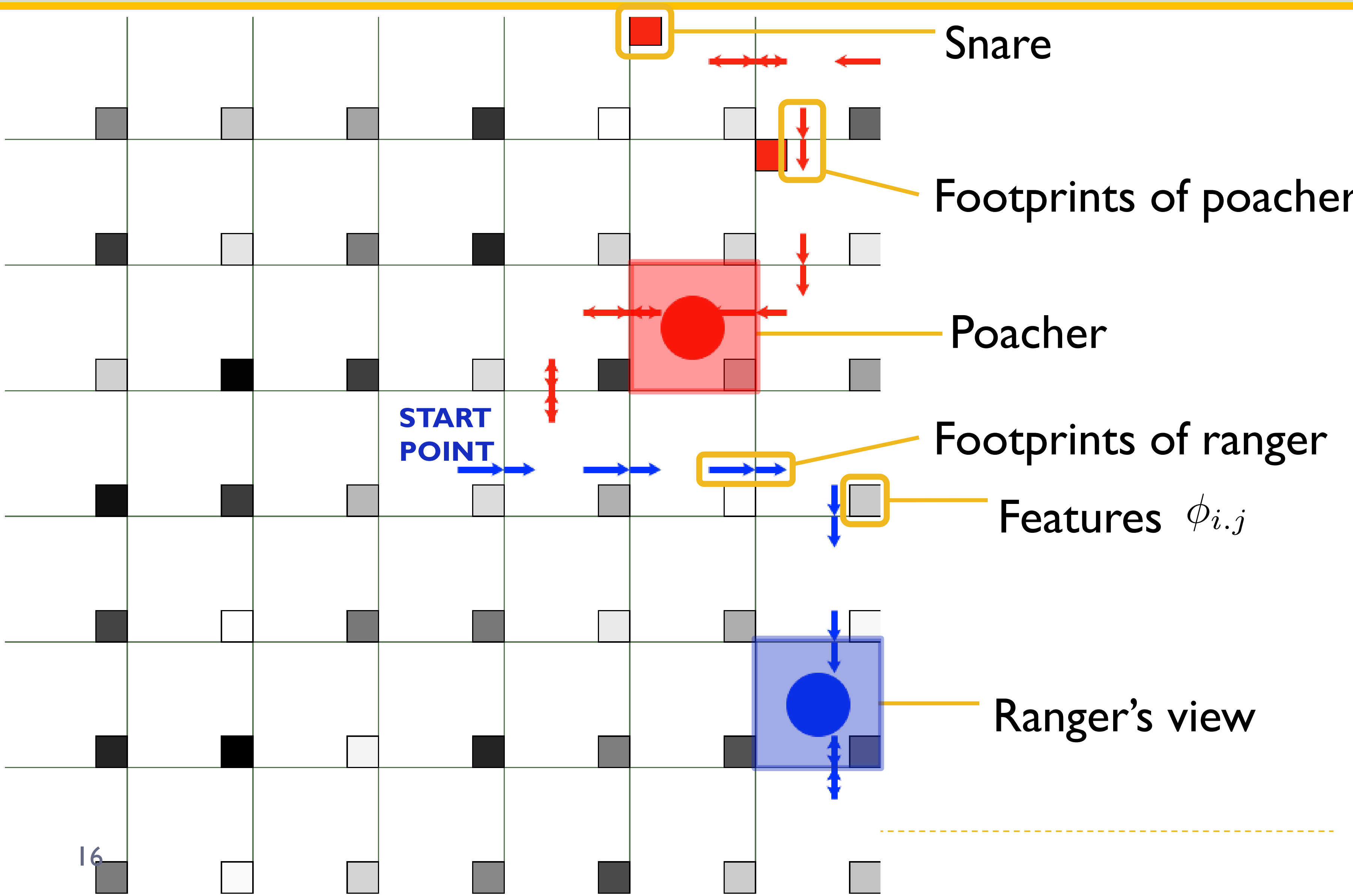
Example: Ranger Patrol with Real-Time Information

- Poachers move in the protected area and place snares (poaching tool)
- Rangers patrol in protected areas to combat poaching
- How should rangers react to real-time information?

Footprints

Deep Reinforcement Learning for Green Security Games with Real-Time Information Yufei Wang, Zheyuan Ryan Shi, Lantao Yu, Yi Wu, Rohit Singh, Lucas Joppa, Fei Fang In AAAI-19

POMDP Model for Ranger Patrol



POMDP Model for Ranger Patrol

POMDP = $(S, A, T, R, \Omega, O, \gamma)$

- Assume the heuristic strategy used by poacher is
  - Randomly move towards one of the 4 neighboring cells
  - Place snare in each cell he visits with probability 0.3
- Assume the ranger gets +3 if catches poacher, +1 if removes snare, and -0.1 for every step regardless
- Ranger can choose to stand still

- Discussion: How to formulate the problem as a POMDP?

POMDP Model for Ranger Patrol

- State:
- Action:
- Transition:
- Reward:
- Observation:
- Observation probability:

POMDP Model for Ranger Patrol

- State: Location of poacher, ranger, snares, footprints in all locations
- Action: Up, Down, Left, Right, Still
- Transition: Determined by poacher's heuristic strategy
- Reward: Determined by the reward rule
- Observation: Location of ranger, footprint, snare
- Observation probability: With prob. 1, observe the ranger location and the footprint/snare in the ranger location

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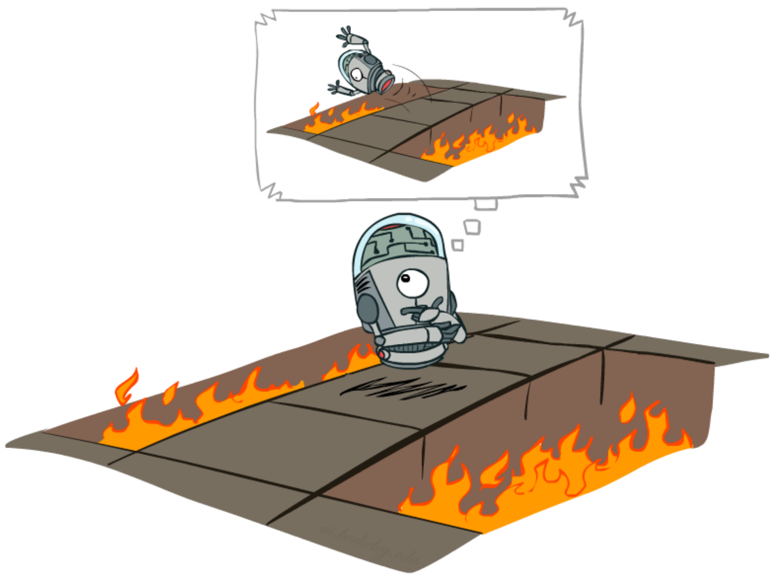
Reinforcement Learning

- Learn an optimal policy for the environment without having a complete model

Don't know T or R (or just hard to enumerate)



Learn through Trial and Error

Don't have a **complete model of the environment**!

Have to actually learn what happens if take an action in a state

Reinforcement Learning

- Learn the optimal policy without knowing T or R

- Model-based RL
  - Build a model (estimate T and R) then find optimal policy

- Model-free RL
  - Directly evaluate without building a model

Model-Based RL with Random Actions

- Choose actions randomly
- Estimate $T(\cdot)$ and $R(\cdot)$ from sample trials (average counts)
- Use estimated $T(\cdot)$ and $R(\cdot)$ to compute estimate of optimal values and optimal policy (i.e., solve the MDP with estimated $T(\cdot)$ and $R(\cdot)$)

Model-Based RL with Random Actions

- Consider a trial

  Start at $(1,1)$

  $s = (1,1)$ action=tright (try going right) based on π

  Reward$= -0.01$; End up at $s' = (2,1)$

  $s = (2,1)$ action=tright (try going right) based on π

  Reward$= -0.01$; End up at $s' = (2,1)$

  $s = (2,1)$ action=tright (try going right) based on π

  Reward$= -0.01$; End up at $s' = (3,1)$

  $s = (3,1)$ action=tright (try going right) based on π

  Reward$= -0.01$; End up at $s' = (4,1)$

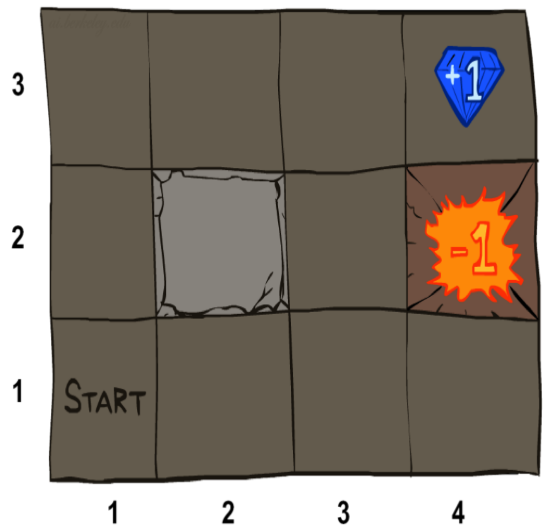
  $s = (4,1)$ action=tup (try going up) based on π

  Reward$= -0.01$; End up at $s' = (4,2)$

  $s = (4,2)$ No action available. Reward$= -1$; Terminate

  Estimate $T\big((2,1)|(2,1)\big)$? $R\big((2,1), tright\big)$?

Environment



Policy: Choose action randomly

Model-Free RL

- Can we find the optimal policy without explicitly estimating $T(\cdot)$ and $R(\cdot)$?

- Value-based method
  - Q-Learning

- Policy-based method + Actor-critic method
  - Policy-gradient
  - Advantage Actor-Critic (A2C)

Q-Learning

Recall $Q^*(s,a) = R(s) + \gamma \sum_{s'} P(s'|s,a) \max_{a'} Q^*(s',a')$

- Q-Learning
  - Start with some random guess of optimal Q values, $\hat{Q}^*(s,a)$
  - Agent interact with the environment following some policy π (no need to be optimal)
  - In one step of a trial: agent is in state s, take action $a = \pi(s)$, get reward r, end up in state s'
  - Update the estimated optimal Q value at (s,a) with

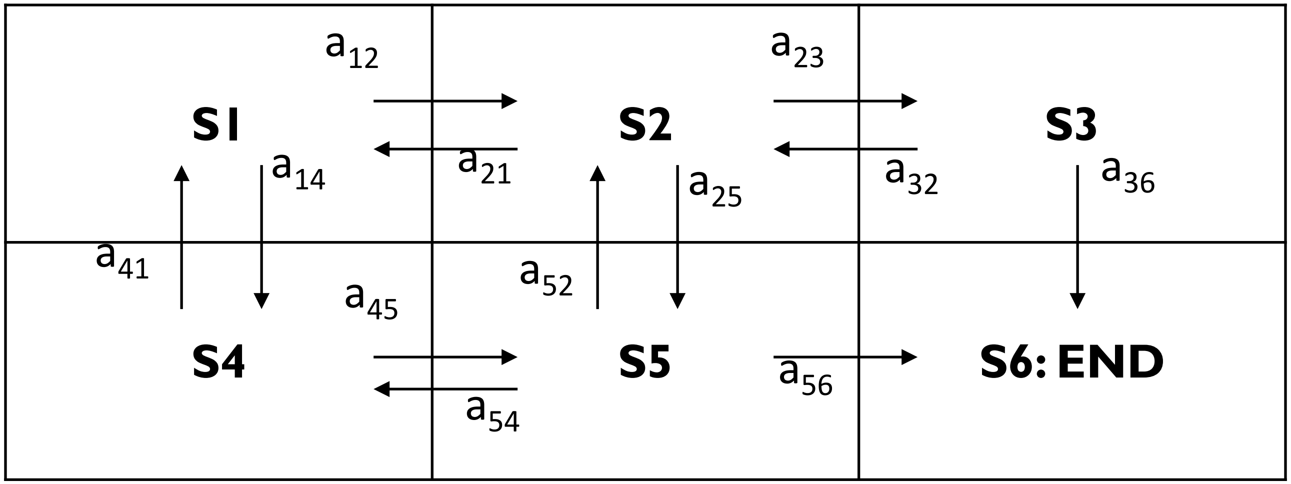
$$\hat{Q}^*(s,a) \leftarrow (1-\alpha)\hat{Q}^*(s,a) + \alpha(r + \gamma \max_{a'} \hat{Q}^*(s',a'))$$

Q-Learning

- Given estimated value of $Q^*(s, a)$, derive estimate of optimal policy

$$\hat{\pi}^*(s) = \underset{a}{\operatorname{argmax}}\, \hat{Q}^*(s, a)$$

Q-Learning Example

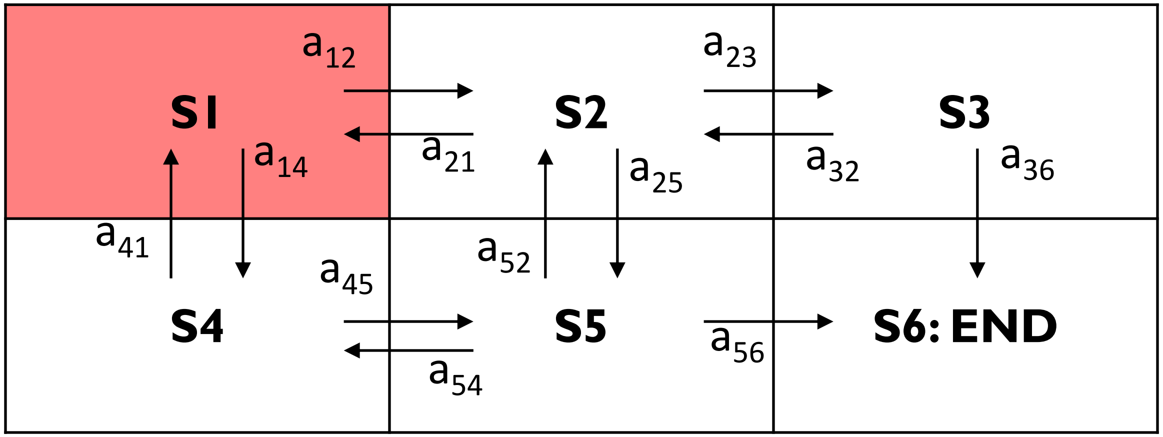


6 states, S1,..S6
12 actions a_{ij}
Deterministic state transitions (but you don't know this beforehand)
$R = 100$ in S6, $R = 0$ otherwise (again, you don't know this)
Use $\gamma = 0.5, \alpha = 1$
Random behavior policy

Q-Learning Example

$$\hat{Q}^*(s,a) \leftarrow (1-\alpha)\hat{Q}^*(s,a) + \alpha\left(r + \gamma \max_{a'} \hat{Q}^*(s',a')\right)$$

$R = 100$ in S6, $\gamma = 0.5, \alpha = 1$



$\hat{Q}^*(S1, a_{12})$	0
$\hat{Q}^*(S1, a_{14})$	0
$\hat{Q}^*(S2, a_{21})$	0
$\hat{Q}^*(S2, a_{25})$	0
$\hat{Q}^*(S2, a_{23})$	0
$\hat{Q}^*(S3, a_{32})$	0
$\hat{Q}^*(S3, a_{36})$	0
$\hat{Q}^*(S4, a_{41})$	0
$\hat{Q}^*(S4, a_{45})$	0
$\hat{Q}^*(S5, a_{52})$	0
$\hat{Q}^*(S5, a_{54})$	0
$\hat{Q}^*(S5, a_{56})$	0
$\hat{Q}^*(S6, null)$	0

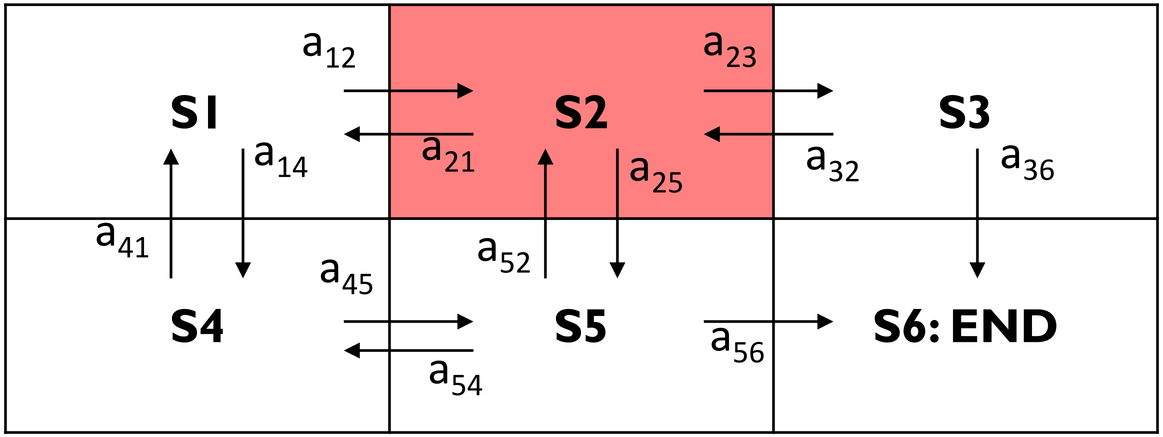
Start at S1, available actions: a_{12}, a_{14}
Probability of choosing each of them: 0.5
Choose a_{12}
Get reward 0, get to state S2
Update state-value function

$$\hat{Q}^*(S1, a_{12}) \leftarrow (1-\alpha)\hat{Q}^*(S1, a_{12}) + \alpha\left(r + \gamma \max_{a' \in \{a_{21}, a_{23}, a_{25}\}} \hat{Q}^*(S2, a')\right) = 0$$

Q-Learning Example

$$\hat{Q}^*(s,a) \leftarrow (1-\alpha)\hat{Q}^*(s,a) + \alpha\left(r + \gamma \max_{a'} \hat{Q}^*(s',a')\right)$$

$R = 100$ in S6, $\gamma = 0.5, \alpha = 1$



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$\hat{Q}^*(S4, a_{45})$	0
$\hat{Q}^*(S5, a_{52})$	0
$\hat{Q}^*(S5, a_{54})$	0
$\hat{Q}^*(S5, a_{56})$	0
$\hat{Q}^*(S6, null)$	0

At S2, available actions: a_{21}, a_{23}, a_{25}

Probability of choosing each of them: $\frac{1}{3}$

Choose a_{23}

Get reward 0, get to state S3

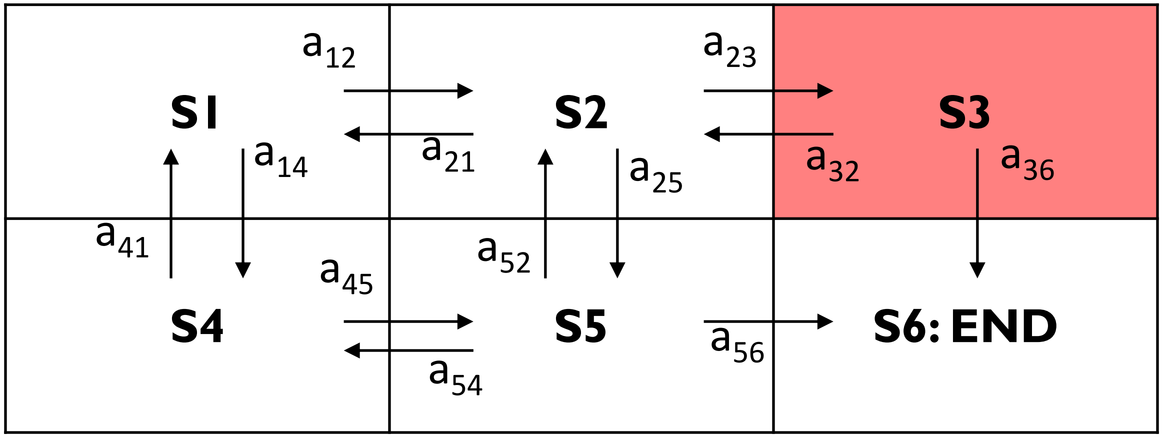
Update state-value function

$$\hat{Q}^*(S2, a_{23}) \leftarrow (1-\alpha)\hat{Q}^*(S2, a_{23}) + \alpha\left(r + \gamma \max_{a' \in \{a_{32}, a_{36}\}} \hat{Q}^*(S3, a')\right) = 0$$

Q-Learning Example

$$\hat{Q}^*(s,a) \leftarrow (1-\alpha)\hat{Q}^*(s,a) + \alpha\left(r + \gamma \max_{a'} \hat{Q}^*(s',a')\right)$$

$R = 100$ in S6, $\gamma = 0.5, \alpha = 1$



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$\hat{Q}^*(S3,a_{32})$	0
$\hat{Q}^*(S3,a_{36})$	0
$\hat{Q}^*(S4,a_{41})$	0
$\hat{Q}^*(S4,a_{45})$	0
$\hat{Q}^*(S5,a_{52})$	0
$\hat{Q}^*(S5,a_{54})$	0
$\hat{Q}^*(S5,a_{56})$	0
$\hat{Q}^*(S6,null)$	0

At S3, available actions: a_{32}, a_{36}
Probability of choosing each of them: 0.5
Choose a_{36}

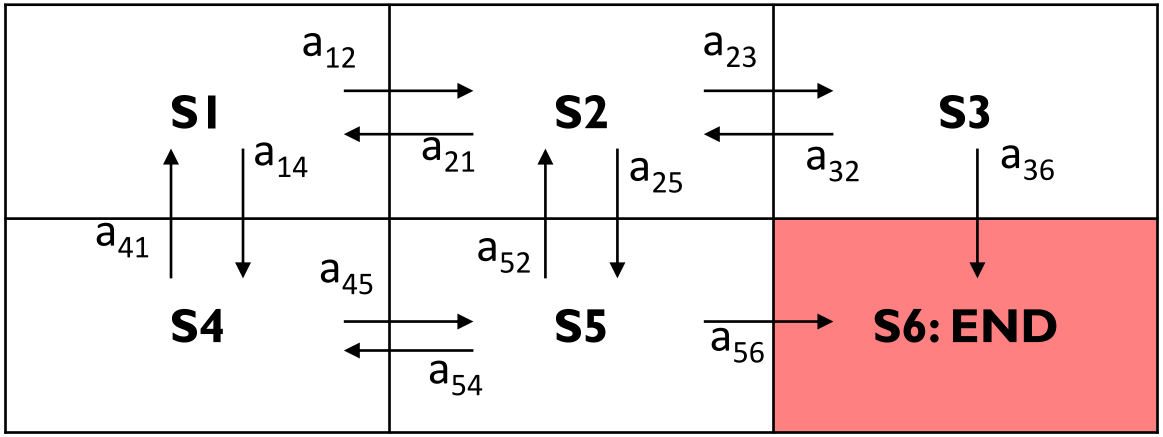
Get reward 0, get to state S6
Update state-value function

$$\hat{Q}^*(S3,a_{36}) \leftarrow (1-\alpha)\hat{Q}^*(S3,a_{36}) + \alpha\left(r + \gamma \max_{a' \in \{null\}} \hat{Q}^*(S6,a')\right) = 0$$

Q-Learning Example

$$\hat{Q}^*(s,a) \leftarrow (1-\alpha)\hat{Q}^*(s,a) + \alpha\left(r + \gamma \max_{a'} \hat{Q}^*(s',a')\right)$$

$R = 100$ in S6, $\gamma = 0.5, \alpha = 1$



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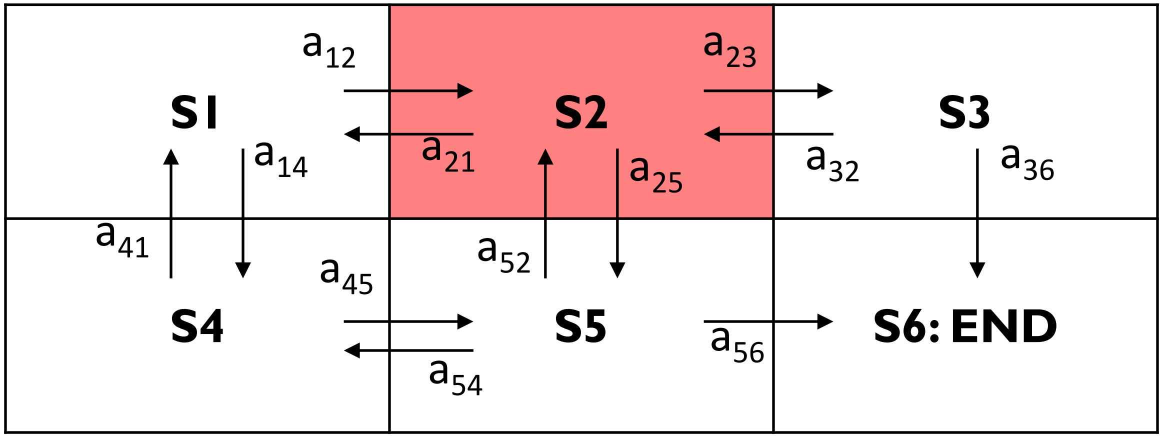
$\hat{Q}^*(S3, a_{36})$	0
$\hat{Q}^*(S4, a_{41})$	0
$\hat{Q}^*(S4, a_{45})$	0
$\hat{Q}^*(S5, a_{52})$	0
$\hat{Q}^*(S5, a_{54})$	0
$\hat{Q}^*(S5, a_{56})$	0
$\hat{Q}^*(S6, null)$	0 $\rightarrow 100$

Terminal state, get reward 100, $\hat{Q}^*(S6, null) \leftarrow 100$

Q-Learning Example

$$\hat{Q}^*(s,a) \leftarrow (1-\alpha)\hat{Q}^*(s,a) + \alpha\left(r + \gamma \max_{a'} \hat{Q}^*(s',a')\right)$$

$R = 100$ in S6, $\gamma = 0.5, \alpha = 1$



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$\hat{Q}^*(S3, a_{36})$	0
$\hat{Q}^*(S4, a_{41})$	0
$\hat{Q}^*(S4, a_{45})$	0
$\hat{Q}^*(S5, a_{52})$	0
$\hat{Q}^*(S5, a_{54})$	0
$\hat{Q}^*(S5, a_{56})$	0
$\hat{Q}^*(S6, null)$	100

Start a new episode!

Start at S2, available actions: a_{21}, a_{23}, a_{25}

Probability of choosing each of them: $\frac{1}{3}$

Choose a_{23}

Get reward 0, get to state S3

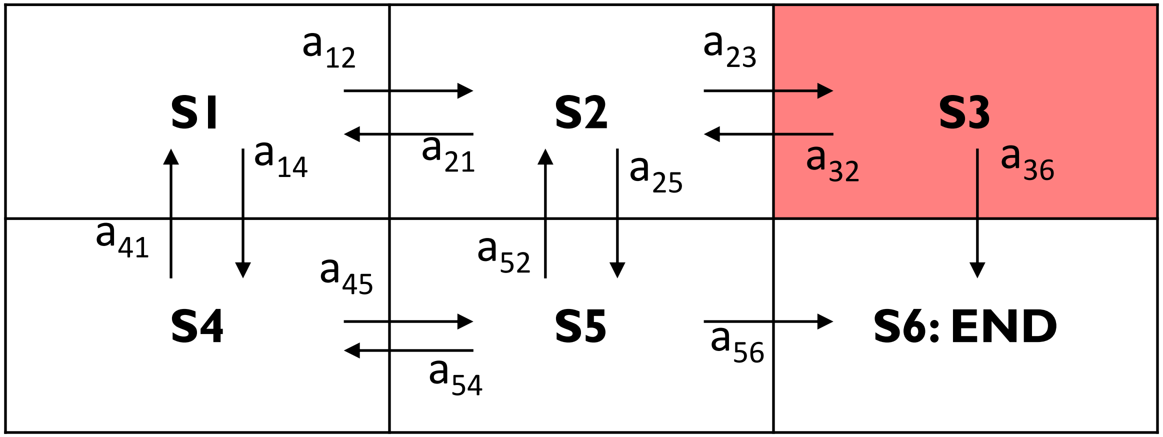
Update state-value function

$$\hat{Q}^*(S2,a_{23}) \leftarrow (1-\alpha)\hat{Q}^*(S2,a_{23}) + \alpha\left(r + \gamma \max_{a' \in \{a_{32},a_{36}\}} \hat{Q}^*(S3,a')\right) = 0$$

Q-Learning Example

$$\hat{Q}^*(s,a) \leftarrow (1-\alpha)\hat{Q}^*(s,a) + \alpha\left(r + \gamma \max_{a'} \hat{Q}^*(s',a')\right)$$

$R = 100$ in S6, $\gamma = 0.5, \alpha = 1$



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$\hat{Q}^*(S2, a_{25})$	0
$\hat{Q}^*(S2, a_{23})$	0
$\hat{Q}^*(S3, a_{32})$	0
$\hat{Q}^*(S3, a_{36})$	0 $\rightarrow 50$
$\hat{Q}^*(S4, a_{41})$	0
$\hat{Q}^*(S4, a_{45})$	0
$\hat{Q}^*(S5, a_{52})$	0
$\hat{Q}^*(S5, a_{54})$	0
$\hat{Q}^*(S5, a_{56})$	0
$\hat{Q}^*(S6, null)$	100

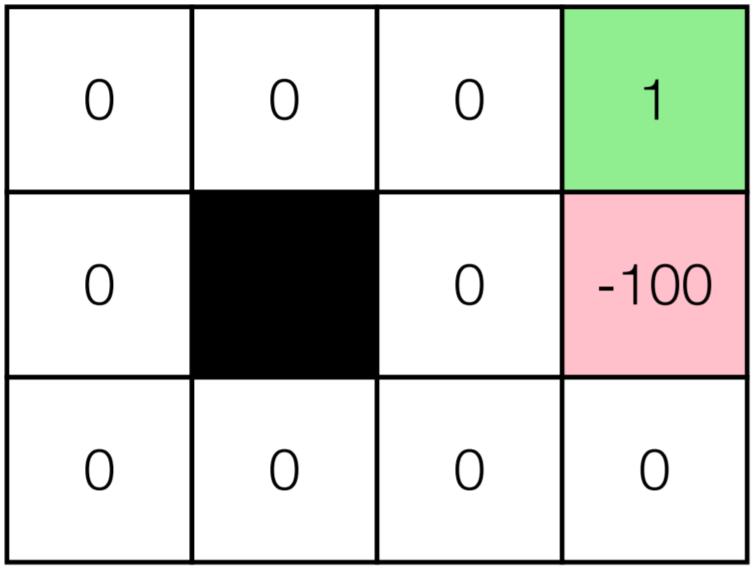
At S3, available actions: a_{32}, a_{36}
Probability of choosing each of them: 0.5
Choose a_{36}

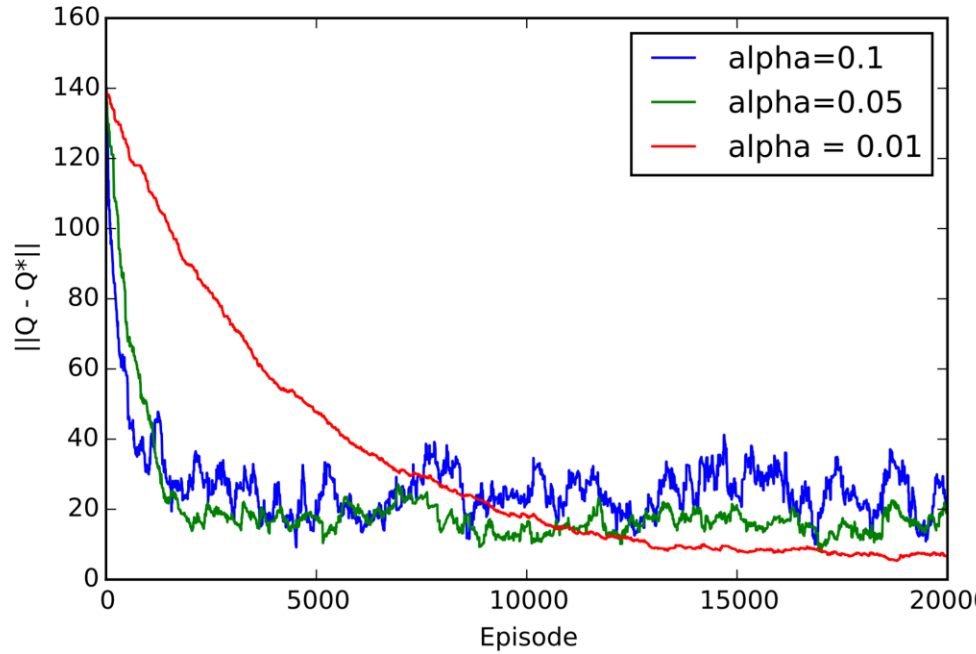
Get reward 0, get to state S6
Update state-value function

$$\hat{Q}^*(S3, a_{36}) \leftarrow (1-\alpha)\hat{Q}^*(S3, a_{36}) + \alpha\left(r + \gamma \max_{a' \in \{null\}} \hat{Q}^*(S6, a')\right) = 50$$

Q-Learning

- Impact of α

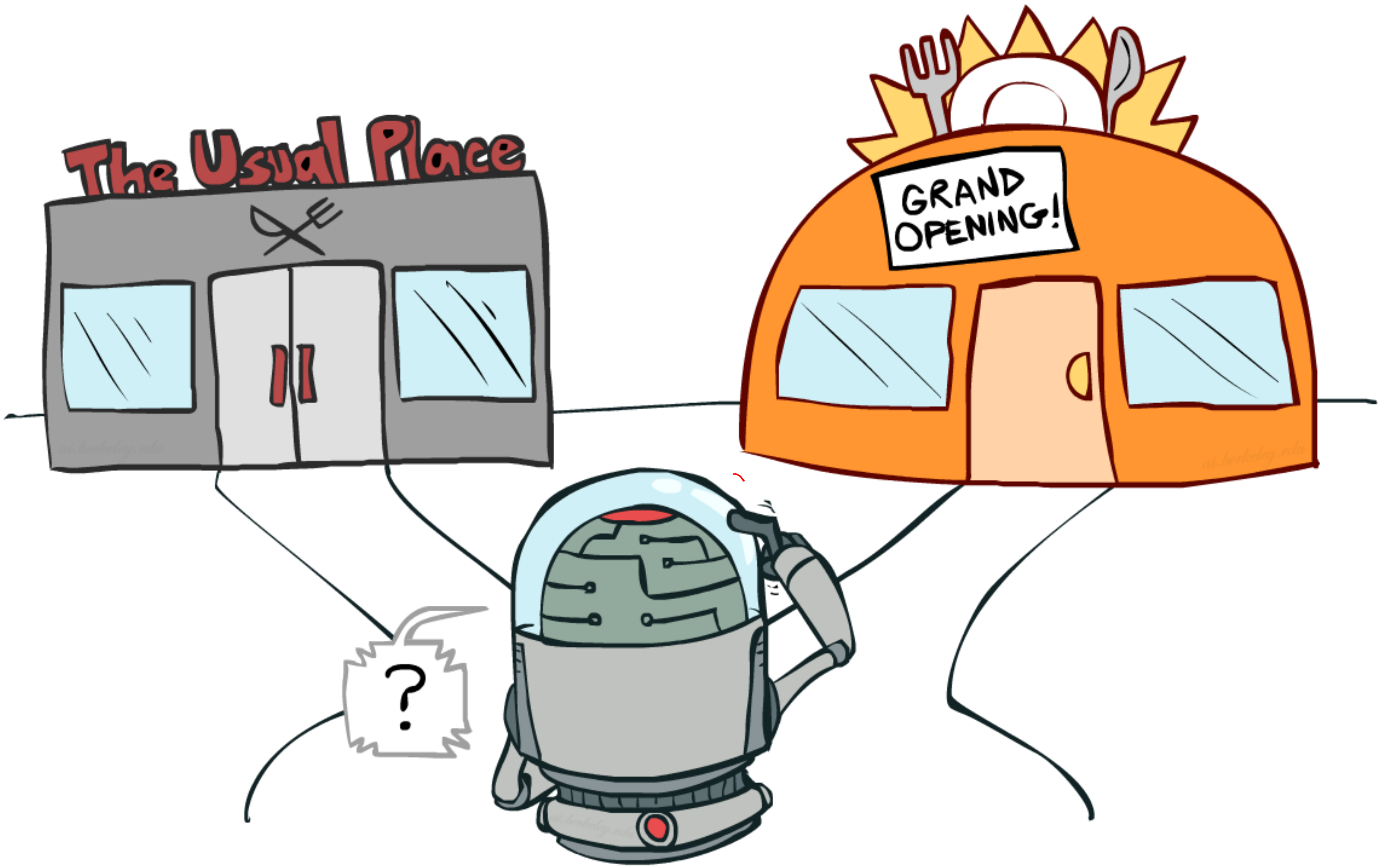




Progress of Q-learning methods for different values of α

- Implication: Let α decrease over time

Exploration vs Exploitation



Simple Approach: ϵ-Greedy

- With probability $1 - \epsilon$
  - Choose action $a = \underset{a'}{\operatorname{argmax}} \hat{Q}^*(s, a')$
- With probability ϵ
  - Select a random action

- For Q-learning
  - Guaranteed to compute optimal policy π^* based on $\hat{Q}^*(s, a)$ given enough samples with $\epsilon > 0$
  - However, the policy the agent is following is never the same as π^* (because it select a random action with probability ϵ)

Policy Gradient

- Key ideas
  - Parameterize the policy
  - Update the parameters towards the direction that increase the objective function (e.g., expected reward)

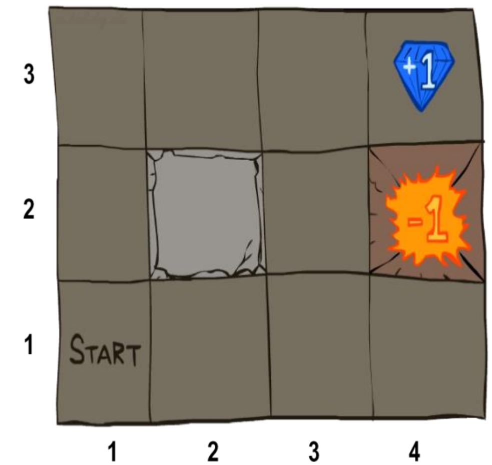
Similar to gradient ascent algorithm

Example

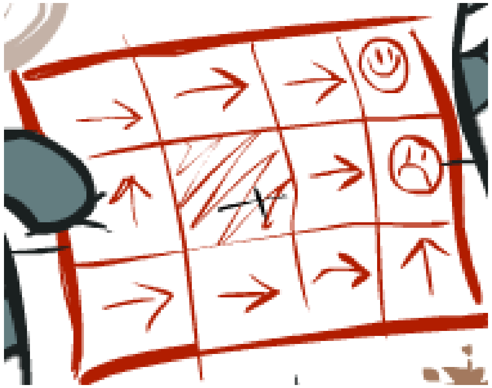
- $\pi = \pi_\theta$ where $\theta \in \mathbb{R}^4$
- $q_k = \dfrac{e^{\theta_k}}{\sum_j e^{\theta_j}}$
- Goal: Find θ that maximizes J_θ (now a standard optimization problem!) where J_θ is often chosen to be $J_\theta = \mathbb{E}_{s \sim \rho_{\pi_\theta}}[V^\pi(s)]$ where ρ_{π_θ} is the stationary distribution
- Policy gradient update (gradient ascent):

$$\theta^{i+1} \leftarrow \theta^i + \alpha \nabla_\theta J_\theta$$

Environment



Policy



Policy Gradient

- Challenge: hard to compute the gradient w.r.t. policy parameters due to uncertainties in MDPs
  - Finite difference methods
  - Policy gradient theorem

Policy Gradient – Policy Gradient Theorem

$$J(\theta) = \mathbb{E}[V^{\pi}(s)]$$

$$\nabla_\theta J(\theta) = \mathbb{E}_{s,a\sim\pi_\theta}[Q^{\pi_\theta}(s,a)\nabla_\theta \log \pi_\theta(s,a)]$$

- Estimate gradient through sampling
  - Sample possible histories
  - Compute gradient as average value of $Q^{\pi_\theta}(s,a)\nabla_\theta \log \pi_\theta(s,a)$
  - How to compute $Q^{\pi_\theta}(s,a)$?
    - REINFORCE: Directly use discounted reward from sampled history

REINFORCE

REINFORCE

Initialize θ arbitrarily
For each episode $\{s_1, a_1, r_2, \dots, s_{T-1}, a_{T-1}, r_T\} \sim \pi_\theta$
For $t = 1 \dots T-1$
$$\theta \leftarrow \theta + \alpha \nabla_\theta \log \pi_\theta(s_t, a_t) v_t$$
(v_t is the discounted reward starting from time t)
Return θ

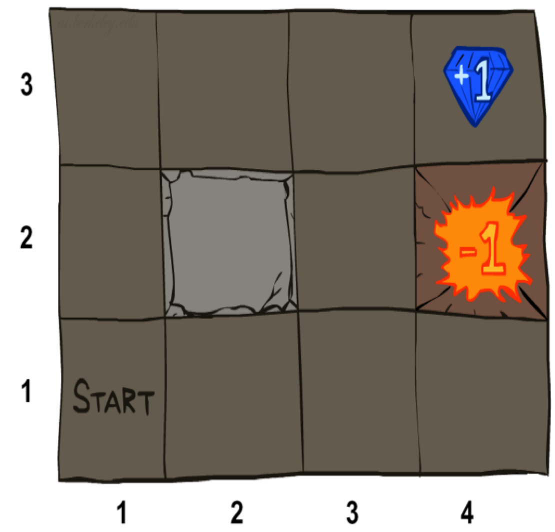
- Similar to stochastic gradient descent
- Use one sample to compute gradient and update parameters

Example

$$q_k = \frac{e^{\theta_k}}{\sum_j e^{\theta_j}}$$

$$\theta \leftarrow \theta + \alpha \nabla_\theta \log \pi_\theta(s_t, a_t) v_t$$

Environment



Start at $(1{,}1)$

$s = (1{,}1)$ action=tright (try going right)

Reward= -0.01; End up at $s' = (2{,}1)$

$$\pi_\theta(s_1, a_1) = \frac{e^{\theta_R}}{e^{\theta_R} + e^{\theta_L} + e^{\theta_U} + e^{\theta_D}}$$

$$\frac{\partial \log \pi_\theta(s_1, a_1)}{\partial \theta_U} = \frac{e^{\theta_R} + e^{\theta_L} + e^{\theta_U} + e^{\theta_D}}{e^{\theta_R}} \times e^{\theta_R} \times (-1)$$

$$\frac{1}{(e^{\theta_R} + e^{\theta_L} + e^{\theta_U} + e^{\theta_D})^2} \times e^{\theta_U} = -\frac{1}{4}$$

$$v_t = -1.05$$

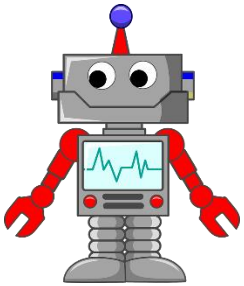
$$\theta_U = \theta_U + (0.5) \times \left(-\frac{1}{4}\right) \times (-1.05)$$

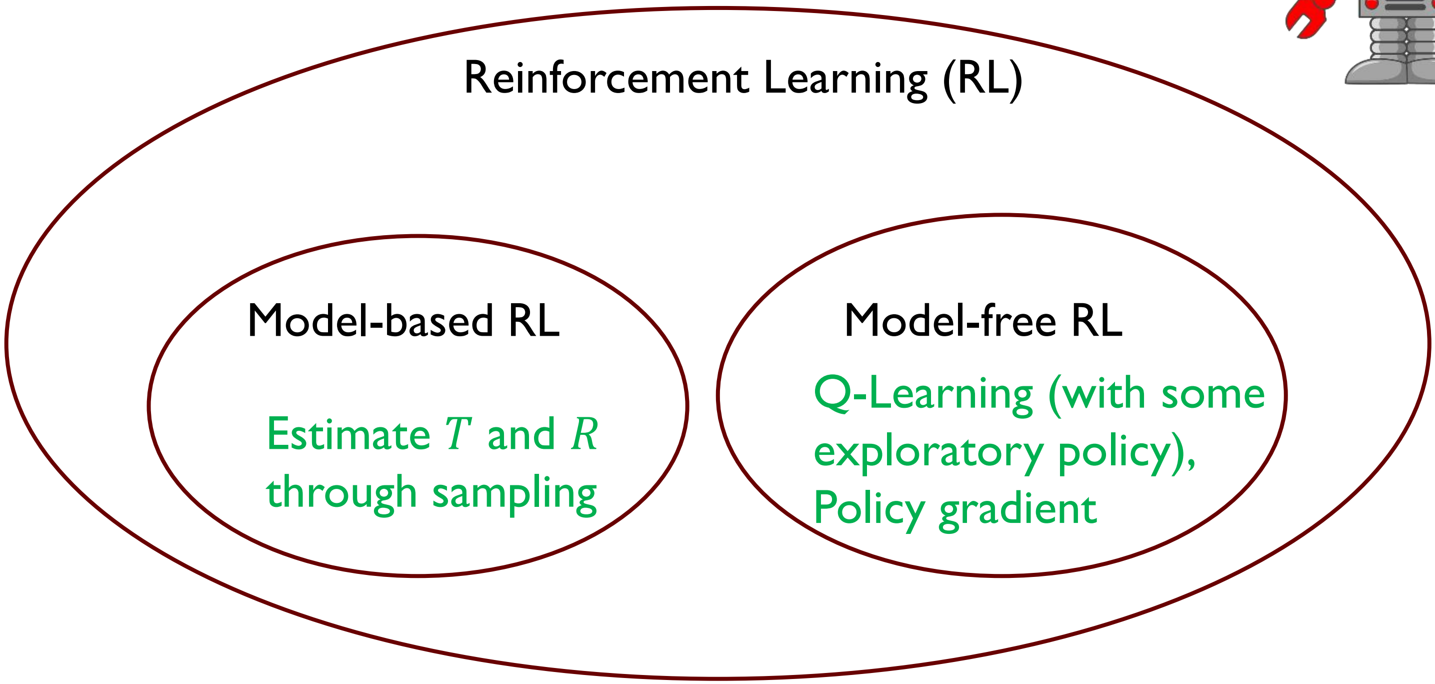
Reinforcement Learning in Practice

- Q-Learning and Proximal Policy Optimization (or PPO, a policy gradient-based approach) are commonly used

- Existing code packages:
  - OpenAI Gym: https://github.com/openai/gym
  - Rllib: https://docs.ray.io/en/latest/rllib/index.html

Summary





Reinforcement Learning (RL)
Model-based RL
Estimate T and R through sampling
Model-free RL
Q-Learning (with some exploratory policy), Policy gradient

References and Additional Resources

Other Resources

- http://courses.csail.mit.edu/6.825/fall05/rl_lecture/rl_examples.pdf
- http://www.cs.cmu.edu/afs/cs/academic/class/15780-s16/www/slides/rl.pdf
- http://incompleteideas.net/book/bookdraft2017nov5.pdf
- https://towardsdatascience.com/a-review-of-recent-reinforcment-learning-applications-to-healthcare-1f8357600407
- http://www0.cs.ucl.ac.uk/staff/d.silver/web/Teaching.html
- https://arxiv.org/pdf/1312.5602.pdf

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Backup Slides

Bellman Equation Explained

- Let $V_t^*(s)$ be the maximal expected total reward assuming
  - We begin in s
  - We have t time steps remaining

- Necessary condition

$$V_t^*(s) = \max_{a \in A} [R(s,a) + \gamma \sum_{s'} \mathbb{P}(s'|s,a) V_{t-1}^*(s')]$$

$$V_0^* = 0$$

- Pick the action which maximizes *current* + *future reward* (assuming continued optimal behavior)

Bellman Equation Explained

$$V_t^*(s) = \max_{a \in A} [R(s,a) + \gamma \sum_{s'} \mathbb{P}(s'|s,a) V_{t-1}^*(s')]$$

- Value function $V^*(s)$ can be viewed as $V_t^*(s)$ as $t \to \infty$

- Bellman Equation
  - $V^*(s) = \max_{a \in A} [R(s,a) + \gamma \sum_{s'} \mathbb{P}(s'|s,a) V^*(s')]$ Fixed point

Value Iteration

Value Iteration

Initialize $V_0^*(s) \leftarrow 0$

Typical termination condition: difference$< \epsilon$

Iterate $V_{i+1}^*(s) \leftarrow \max_{a\in A} [R(s,a) + \gamma \sum_{s'} \mathbb{P}(s'|s,a) V_i^*(s')]$

Based on state-value Bellman Equations

$$V^*(s) = \max_{a\in A} [R(s,a) + \gamma \sum_{s'} \mathbb{P}(s'|s,a) V^*(s')]$$

- Consider the finite horizon view: Pretend we have a really long horizon
- aka 'Value update', 'Bellman backups/updates'
- Guaranteed to converge to $V^*(s)$

What is the optimal policy π^*?

Policy Iteration

- Not necessary to get an accurate estimate of V^* to induce π^*

- Policy iteration
  - Compute optimal policy π^* directly
  - Iterate between 2 steps
    - Policy Evaluation (check how good current policy is)
    - Policy Improvement (get a 'better' policy)

Policy Iteration

- Policy Evaluation
  - Method 1: Iterative approach
    - $V_{i+1}^{\pi}(s) \leftarrow R(s,\pi(s)) + \gamma \sum_{s'} \mathbb{P}(s'|s,a) V_i^{\pi}(s')\,, V_0^{\pi}(s) \leftarrow 0$
  - Method 2: Solve system of linear equations
    - $V^{\pi}(s) = R(s,\pi(s)) + \gamma \sum_{s'} \mathbb{P}(s'|s,a) V^{\pi}(s')$
- Policy Improvement
  - $\pi^{new}(s) \leftarrow \underset{a\in A}{\operatorname{argmax}} \left[R(s,a) + \gamma \sum_{s'} \mathbb{P}(s'|s,a) V^{\pi}(s')\right]$
  - Note that when π is optimal, π^{new} is the same as π
- Policy iteration converges to the optimal policy in a finite number of steps
- Often converge faster than value iteration

Policy Gradient – Policy Gradient Theorem

- Basis: Given function $f(\cdot)$ and discrete-valued random variable $\boldsymbol{X} \sim p(\boldsymbol{x}|\boldsymbol{\theta})$

$$\nabla_{\boldsymbol{\theta}} \mathbb{E}_{\boldsymbol{X}}[f(\boldsymbol{X})] = \mathbb{E}_{\boldsymbol{X}}[\overbrace{\boxed{f(\boldsymbol{X}) \nabla_{\boldsymbol{\theta}} \log p(\boldsymbol{X}|\boldsymbol{\theta})}}^{g(X)}]$$

- Can be approximated by sampling X and compute average $g(X)$!

Policy Gradient – Policy Gradient Theorem

- Basis: Given function $f(\cdot)$ and discrete-valued random variable $\boldsymbol{X} \sim p(\boldsymbol{x}|\boldsymbol{\theta})$

$$\nabla_{\boldsymbol{\theta}} \mathbb{E}_{\boldsymbol{X}}[f(\boldsymbol{X})] = \mathbb{E}_{\boldsymbol{X}}[\overset{g(X)}{\boxed{f(\boldsymbol{X}) \nabla_{\boldsymbol{\theta}} \log p(\boldsymbol{X}|\boldsymbol{\theta})}}]$$

Derivation: $\nabla_{\boldsymbol{\theta}} \mathbb{E}_{\boldsymbol{X}}[f(\boldsymbol{X})] = \nabla_{\boldsymbol{\theta}} \sum_{\boldsymbol{x}} p(\boldsymbol{x}|\boldsymbol{\theta}) f(\boldsymbol{x}) = \sum_{\boldsymbol{x}} f(\boldsymbol{x}) \nabla_{\boldsymbol{\theta}} p(\boldsymbol{x}|\boldsymbol{\theta})$

$$= \sum_{\boldsymbol{x}} f(\boldsymbol{x}) p(\boldsymbol{x}|\boldsymbol{\theta}) \frac{\nabla_{\boldsymbol{\theta}} p(\boldsymbol{x}|\boldsymbol{\theta})}{p(\boldsymbol{x}|\boldsymbol{\theta})}$$

$$= \sum_{\boldsymbol{x}} f(\boldsymbol{x}) p(\boldsymbol{x}|\boldsymbol{\theta}) \nabla_{\boldsymbol{\theta}} \log p(\boldsymbol{x}|\boldsymbol{\theta})$$

$$= \mathbb{E}_{\boldsymbol{X}}[f(\boldsymbol{X}) \nabla_{\boldsymbol{\theta}} \log p(\boldsymbol{X}|\boldsymbol{\theta})]$$

- Can be approximated by sampling X and compute average $g(X)$!

Policy Gradient – Policy Gradient Theorem

- Given $\nabla_{\boldsymbol{\theta}} \mathbb{E}_{\boldsymbol{X}}[f(\boldsymbol{X})] = \mathbb{E}_{\boldsymbol{X}}[f(\boldsymbol{X}) \nabla_{\boldsymbol{\theta}} \log p(\boldsymbol{X}|\boldsymbol{\theta})]$, rewrite the gradient of the objective function $J(\theta)$ with respect to policy parameters

$$\nabla_\theta J(\theta) = \nabla_\theta \mathbb{E}_{s\sim\pi_\theta}[V^\pi(s)] = \nabla_\theta \mathbb{E}_{s\sim\pi_\theta}\left[\sum_a \pi_\theta(s,a) Q^{\pi_\theta}(s,a)\right]$$

$$\nabla_\theta J(\theta) = \mathbb{E}_{s,a\sim\pi_\theta}[Q^{\pi_\theta}(s,a) \nabla_\theta \log \pi_\theta(s,a)]$$

- Estimate gradient through sampling
  - Sample possible histories
  - Compute gradient as average value of $Q^{\pi_\theta}(s,a) \nabla_\theta \log \pi_\theta(s,a)$
  - How to compute $Q^{\pi_\theta}(s,a)$?
    - REINFORCE: Directly use discounted reward from sampled history

Example

$$q_k = \frac{e^{\theta_k}}{\sum_j e^{\theta_j}} \qquad \theta \leftarrow \theta + \alpha \nabla_\theta \log \pi_\theta(s_t, a_t) v_t$$

- Consider a trial

Start at $(1,1)$

$s = (1,1)$ action=tright (try going right)

Reward= -0.01; End up at $s' = (2,1)$

$s = (2,1)$ action=tright (try going right)

Reward= -0.01; End up at $s' = (2,1)$

$s = (2,1)$ action=tright (try going right)

Reward= -0.01; End up at $s' = (3,1)$

$s = (3,1)$ action=tright (try going right)

Reward= -0.01; End up at $s' = (4,1)$

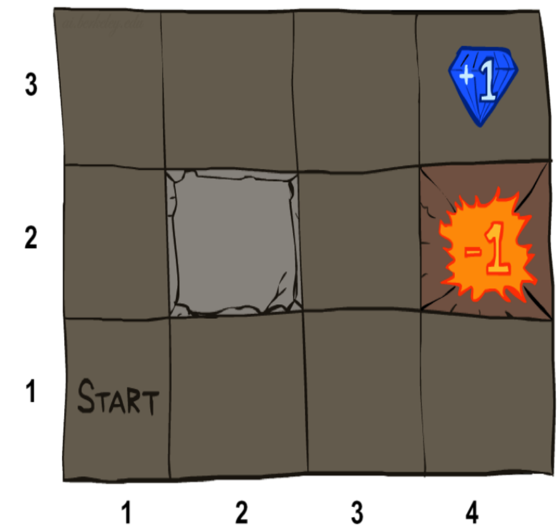
$s = (4,1)$ action=tup (try going up)

Reward= -0.01; End up at $s' = (4,2)$

$s = (4,2)$ No action available. Reward= -1; Terminate

$\gamma = 1, \alpha = 0.5, \theta$ is initialized to 0, how to update θ?

Environment



Example

$$q_k = \frac{e^{\theta_k}}{\sum_j e^{\theta_j}} \qquad \theta \leftarrow \theta + \alpha \nabla_\theta \log \pi_\theta(s_t, a_t) v_t$$

Start at $(1,1)$

$s = (1,1)$ action=tright (try going right)

Reward= -0.01; End up at $s' = (2,1)$

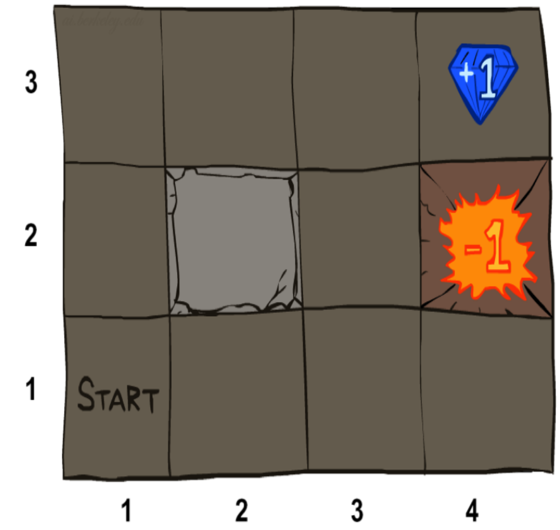
$\pi_\theta(s_1, a_1) =$

$\frac{\partial \log \pi_\theta(s_1, a_1)}{\partial \theta_U} =$

$v_t =$

$\theta_U = \theta_U +$

Environment



Discussion: How is it different for the update of θ_R?