Reminder

- HW4 due 3/21
- Course project progress report 2 due 3/26
- PRA5 due 3/28
- Come to OH for course project discussion!

Artificial Intelligence Methods for Social Good

Lecture 18:

Basics of Game Theory

17-537 (9-unit) and 17-737 (12-unit)

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Learning Objectives

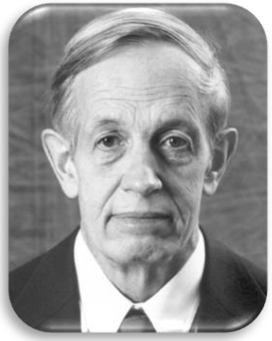
- Understand the concept of
  - Game, Player, Action, Strategy, Payoff, Expected utility, Best response
  - Maxmin Strategy, Minmax Strategy
  - Nash Equilibrium
- Write down the linear program for finding maxmin/minmax strategy
- Describe Minimax Theory
- For the ferry protection problem, briefly describe
  - Significance/Motivation
  - Task being tackled, i.e., what is being solved/optimized
  - Model and method used to solve the problem
  - Evaluation process and criteria

From Games to Game Theory

- The study of mathematical models of conflict and cooperation between intelligent decision makers
- Used in economics, political science etc

John von Neumann

John Nash



Heinrich Freiherr von Stackelberg

Winners of Nobel Memorial Prize in Economic Sciences

Outline

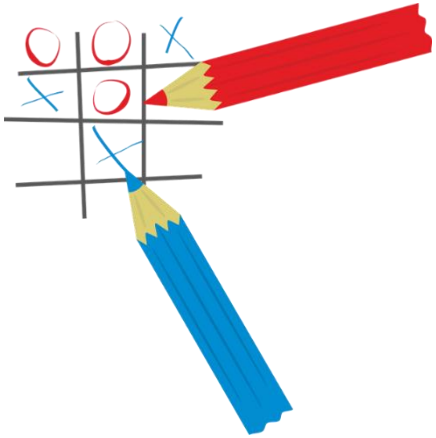
- Normal-Form Games
- Solution Concepts
- Ferry Protection

Some Classical Games

- Rock-Paper-Scissors (RPS)
  - Rock beats Scissors
  - Scissors beats Paper
  - Paper beats Rock
- Prisoner's Dilemma (PD)
  - If both Cooperate: 1 year in jail each
  - If one Defect, one Cooperate: 0 year for (D), 3 years for (C)
  - If both Defect: 2 years in jail each

Some Classical Games

- Football vs Concert (FvsC)
  - Historically known as Battle of Sexes
  - If football together: Alex ☺☺, Berry ☺
  - If concert together: Alex ☺, Berry ☺☺
  - If not together: Alex ☹, Berry ☹
- Tic-Tac-Toe (TTT)



Normal-Form Games

- A finite, n-player normal-form game is described by a tuple (N, A, u)
  - Set of players $N = \{1..n\}$
  - Set of joint actions $A = \prod_i A_i$
    - $\mathbf{a} = (a_1, \dots, a_n) \in A$ is an action profile
  - Payoffs / Utility functions $u_i: A \rightarrow \mathbb{R}$
    - $u_i(a_1, \dots, a_n)$ or $u_i(\mathbf{a})$
- Players move simultaneously and then game ends immediately
- Zero-Sum Game: $\sum_i u_i(\mathbf{a}) = 0, \forall \mathbf{a}$

May also be called matrix form, strategic form, or standard form

Payoff Matrix

- A two-player normal-form game with finite actions can be represented by a (bi)matrix
  - Player 1: Row player, Player 2: Column player
  - First number is the utility for Player 1, second for Player 2

Player 2

Player 1

	Rock	Paper	Scissors
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissor	-1,1	1,-1	0,0

Player 2

Player 1

	Cooperate	Defect
Cooperate	-1,-1	-3,0
Defect	0,-3	-2,-2

Berry

Alex

	Football	Concert
Football	2,1	0,0
Concert	0,0	1,2

Q: What if we have more than 2 players?

Pure Strategy, Mixed Strategy, Support

- Pure strategy: choose an action deterministically
- Mixed strategy: choose action randomly
- Given action set A_i, player i's strategy set is $S_i = \Delta^{|A_i|}$

- Support: set of actions chosen with non-zero probability

Expected Utility

- Given players' strategy profile $\mathbf{s} = (s_1, \dots, s_n)$, what is the expected utility for each player?
- Let $s_i(a)$ be the probability of choosing action $a \in A_i$, then
  - $u_i(s_1, \dots, s_n) =$

Expected Utility

- Given players' strategy profile $\mathbf{s} = (s_1, \dots, s_n)$, what is the expected utility for each player?
- Let $s_i(a)$ be the probability of choosing action $a \in A_i$, then
  - $u_i(s_1, \dots, s_n) = \sum_{\mathbf{a} \in \mathrm{A}} P(\mathbf{a}) u_i(\mathbf{a}) = \sum_{\mathbf{a} \in \mathrm{A}} u_i(\mathbf{a}) \prod_{i'} s_{i'}(a_{i'})$

Best Response

- Let $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots a_n)$.
- An action profile can be denoted as $\mathbf{a} = (a_i, a_{-i})$
- Similarly, define u_{-i} and s_{-i}

- Best Response: Set of actions or strategies leading to highest expected utility given the strategies or actions of other players
  - $a_i^* \in BR(a_{-i})$ iff
  - $s_i^* \in BR(s_{-i})$ iff

- Theorem (Nash 1951): A mixed strategy is BR iff all actions in the support are BR
  - $s_i \in BR(s_{-i})$ iff

Best Response

- Let $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots a_n)$.
- An action profile can be denoted as $\mathbf{a} = (a_i, a_{-i})$
- Similarly, define u_{-i} and s_{-i}

- Best Response: Set of actions or strategies leading to highest expected utility given the strategies or actions of other players
  - $a_i^* \in BR(a_{-i})$ iff $\forall a_i \in A_i, u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i})$
  - $s_i^* \in BR(s_{-i})$ iff $\forall s_i \in S_i, u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$

- Theorem (Nash 1951): A mixed strategy is BR iff all actions in the support are BR
  - $s_i \in BR(s_{-i})$ iff $\forall a_i: s_i(a_i) > 0, a_i \in BR(s_{-i})$

Outline

- Normal-Form Games
- Solution Concepts
- Ferry Protection

Nash Equilibrium

- Nash Equilibrium (NE)
  - $\mathbf{s} = \langle s_1, \dots, s_n \rangle$ is NE if $\forall i, s_i \in BR(s_{-i})$
  - Everyone's strategy is a BR to others' strategy profile
  - Focus on strategy profile for all players
  - One cannot gain by unilateral deviation
  - Pure Strategy Nash Equilibrium (PSNE)
    - $\mathbf{a} = \langle a_1, \dots, a_n \rangle$ is PSNE if $\forall i, a_i \in BR(a_{-i})$
  - Mixed Strategy NE: at least one player use a mixed strategy

Player 2

Player 1

	Cooperate	Defect
Cooperate	-1,-1	-3,0
Defect	0,-3	-2,-2

Q1: What are the PSNEs in this game?

Q2: Given a mixed strategy, how to determine whether it is an NE?

Q3: Is $\left(\frac{1}{2},\frac{1}{2}\right),(\frac{1}{2},\frac{1}{2})$ an NE for this game?

Poll 1

$\mathbf{s} = \langle s_1, \dots, s_n \rangle$ is NE if $\forall i, s_i \in BR(s_{-i})$

Is the following strategy profile an NE?
Alex: (2/3,1/3), Berry: (1/3,2/3)

A: Yes
B: No
C: I don't know

		Berry	
		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2

Poll 1

$\mathbf{s} = \langle s_1, \dots, s_n \rangle$ is NE if $\forall i, s_i \in BR(s_{-i})$

Is the following strategy profile an NE?
Alex: (2/3,1/3), Berry: (1/3,2/3)

$$u_A(s_A, s_B) = \frac{2}{3} * \frac{1}{3} * 2 + \frac{1}{3} * \frac{2}{3} * 1 = 2/3$$

$$u_A(F, s_B) = 2 * \frac{1}{3} = \frac{2}{3}$$

$$u_A(C, s_B) = 1 * \frac{2}{3} = \frac{2}{3}$$

So $u_A(s_A', s_B) = \epsilon u_A(F, s_B) + (1 - \epsilon) u_A(C, s_B) = 2/3$

So Alex has no incentive to deviate (u_A cannot increase)

Similar reasoning goes for u_B

Berry

Alex

	Football	Concert
Football	2,1	0,0
Concert	0,0	1,2

Nash Equilibrium

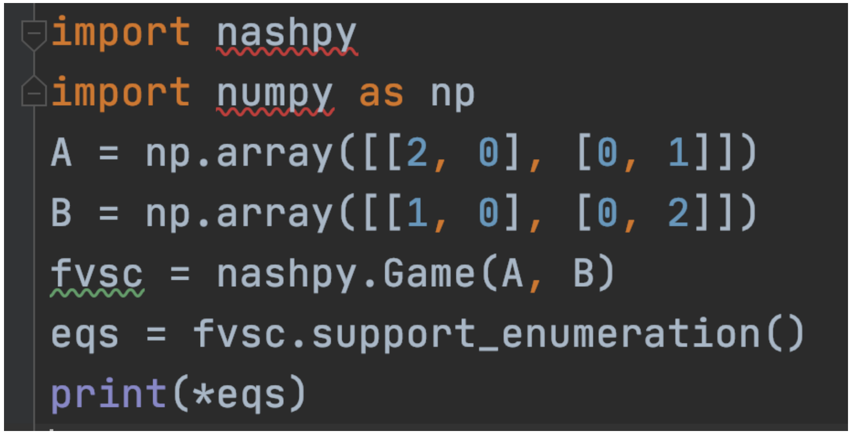
- Theorem (Nash 1951): NE always exists in finite games
  - Finite game: $n < \infty, |A| < \infty$
  - NE: pure or mixed

Compute Nash Equilibrium

- Find all Nash Equilibrium (two-player)
  - Support Enumeration Method
  - Lemke-Howson Algorithm
    - Linear Complementarity Programming (LCP) formulation
    - Solve by pivoting on support (similar to Simplex algorithm)
  - In practice, available solvers/packages: Nashpy (python), gambit project

		Berry	
		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2

With Nashpy 0.0.19



```
import nashpy
import numpy as np
A = np.array([[2, 0], [0, 1]])
B = np.array([[1, 0], [0, 2]])
fvsc = nashpy.Game(A, B)
eqs = fvsc.support_enumeration()
print(*eqs)
```

Maximin Strategy

- Maximin Strategy (applicable to multiplayer games)
  - Maximize worst case expected utility
  - Maximin strategy for player i is $\arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$
  - Maximin value for player i is $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$
    (Also called safety level)
  - Focus on single player's strategy
  - Can be computed through linear programming

Compute Maximin Strategy

- For bimatrix games, maximin strategy can be computed through linear programming
- Let U_{ij}^{1} be player 1's payoff value when player 1 choose action i and player 2 choose action j

Denote $s_1 = \langle x_1, \ldots, x_{|A_1|} \rangle$ where x_i is the probability of choosing the i^{th} action of player 1

Compute Maximin Strategy

- For bimatrix games, maximin strategy can be computed through linear programming
- Let U_{ij}^1 be player 1's payoff value when player 1 choose action i and player 2 choose action j

To get $\underset{s_1}{\operatorname{argmax}} \min_{s_2} u_1(s_1, s_2)$, we denote $s_1 = \langle x_1, \dots, x_{|A_1|} \rangle$ where x_i is the probability of choosing the i^{th} action of player 1. Now we need to find the value of x_i

$$\max_{x_1,\dots,x_{|A_1|}} \min_j \sum_i x_i U_{ij}^1$$

$$\text{s.t.} \sum_i x_i = 1$$

$$x_i \geq 0$$

Only need to check pure strategies. Recall the theorem of BR: A mixed strategy is BR iff all actions in the support are BR

Compute Maximin Strategy

- Convert to LP

$\mathcal{P}_1$

$$\max_x \min_j \sum_i x_i U_{ij}^1$$
$$\text{s.t. } \textstyle\sum_i x_i = 1$$
$$x_i \geq 0$$

$\mathcal{P}_2$ -- LP

$$\max_{x,v} v$$
$$\text{s.t. } v \leq \textstyle\sum_i x_i U_{ij}^1, \forall j$$
$$\sum_i x_i = 1$$
$$x_i \geq 0$$

- Claim: x^* is optimal solution for $\mathcal{P}_1$ iff it is optimal solution for $\mathcal{P}_2$

Compute Maximin Strategy

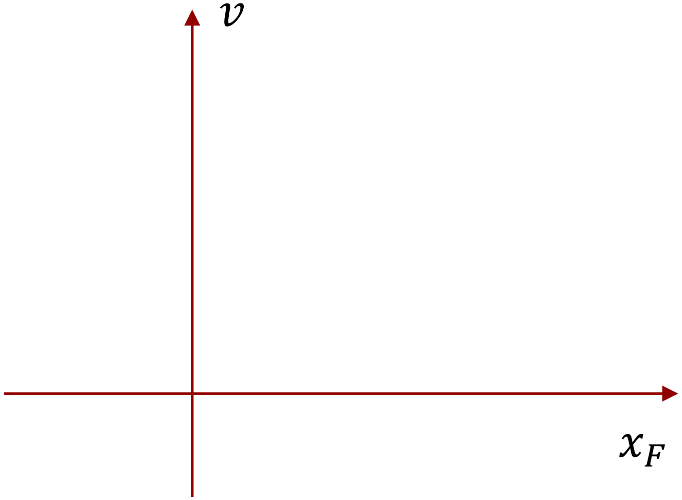
$$\max_{x,v} v$$

$$\text{s.t. } v \leq \sum_i x_i U^1_{ij}, \forall j$$

$$\sum_i x_i = 1$$

$$x_i \geq 0$$

		Berry	
		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2



Compute Maximin Strategy

$$\max_{x,v} v$$
$$\text{s.t. } v \leq \sum_i x_i U^1_{ij}, \forall j$$
$$\sum_i x_i = 1$$
$$x_i \geq 0$$

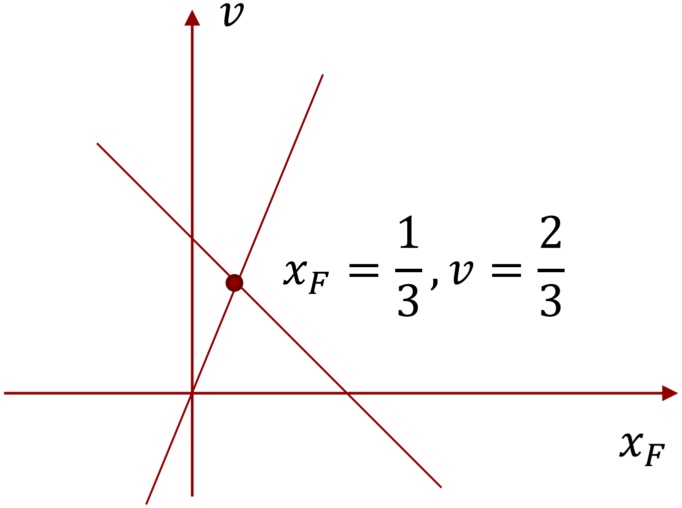
$$\max_{x_F, x_C, v} \ v$$
$$\text{s.t. } v \leq x_F * 2 + x_C * 0$$
$$v \leq x_F * 0 + x_C * 1$$
$$x_F + x_C = 1$$
$$x_F \geq 0, x_C \geq 0$$

$$\max_{x_F, v} v$$
$$\text{s.t. } v \leq 2x_F$$
$$v \leq 1 - x_F$$
$$0 \leq x_F \leq 1$$

Berry

Alex

	Football	Concert
Football	2,1	0,0
Concert	0,0	1,2



Minimax Strategy

- Minimax Strategy in two-player games:
  - Minimize best case expected utility for the other player (just want to harm your opponent)
  - Minimax strategy for player i against player $-i$ is $\operatorname*{argmin}_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$
  - Minimax value for player $-i$ is $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$
  - Focus on single player's strategy
  - Can be computed through linear programming

Compute Minimax Strategy

$\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$

- Minimax strategy can be found through LP
- Let U_{ij}^2 be player 2's payoff value when player 1 choose action i and player 2 choose action j
- Let $s_1 = \langle x_1, \dots, x_{|A_1|} \rangle$ where x_i is the probability of choosing the i^{th} action of player 1

$$\min_{x,v} v$$

$$\text{s.t. } v \geq \textstyle\sum_i x_i U_{ij}^2, \forall j$$

$$\sum_i x_i = 1$$

$$x_i \geq 0$$

Compute Minimax Strategy

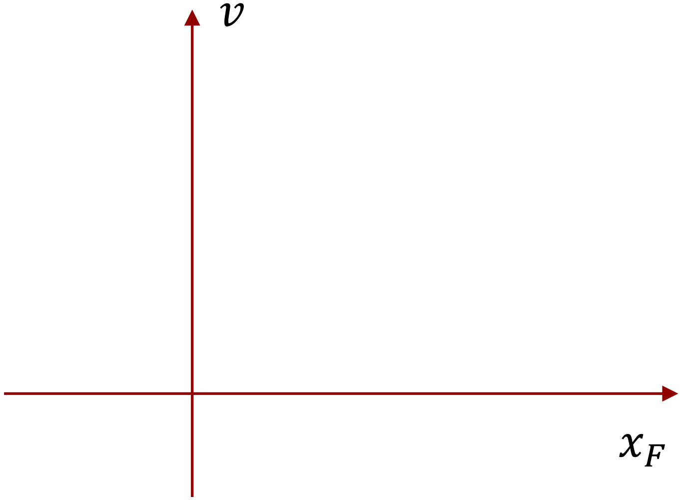
$$\min_{x,v} v$$

s.t. $v \geq \sum_i x_i U_{ij}^2, \forall j$

$$\sum_i x_i = 1$$

$$x_i \geq 0$$

		Berry	
		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2



Compute Minimax Strategy

$$\min_{x,v} v$$
$$\text{s.t. } v \geq \sum_i x_i U_{ij}^2, \forall j$$
$$\sum_i x_i = 1$$
$$x_i \geq 0$$

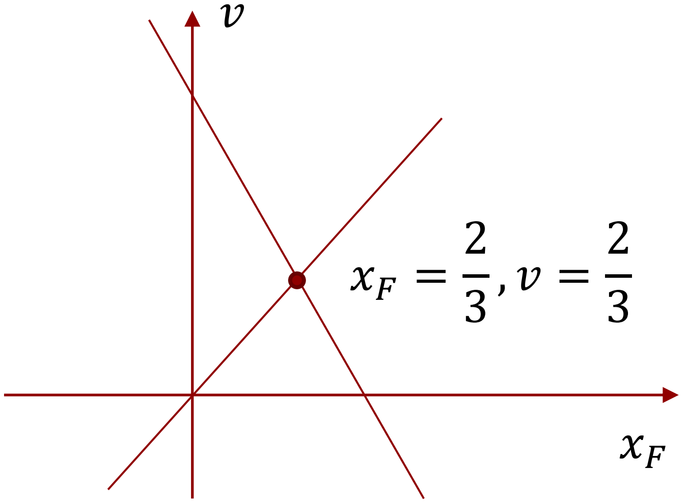
➡

$$\min_{x_F, x_C, v} \; v$$
$$\text{s.t. } v \geq x_F * 1 + x_C * 0$$
$$v \geq x_F * 0 + x_C * 2$$
$$x_F + x_C = 1$$
$$x_F \geq 0, x_C \geq 0$$

➡

$$\min_{x_F, v} v$$
$$\text{s.t. } v \geq x_F$$
$$v \geq 2(1 - x_F)$$
$$0 \leq x_F \leq 1$$

		Berry	
		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2



$$x_F = \frac{2}{3}, v = \frac{2}{3}$$

Minimax Theorem

- Theorem (von Neumann 1928, Nash 1951):
  - Informal: Minimax value=Maximin value=NE value in finite 2-player zero-sum games
  - Formally
    - $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = \min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$
    - $\exists v \in \mathbb{R}$ such that Player 1 can guarantee value at least v and Player 2 can guarantee loss at most v (v is called value of the game)
  - Indication: All NEs leads to the same utility profile in a finite two-player zero-sum game

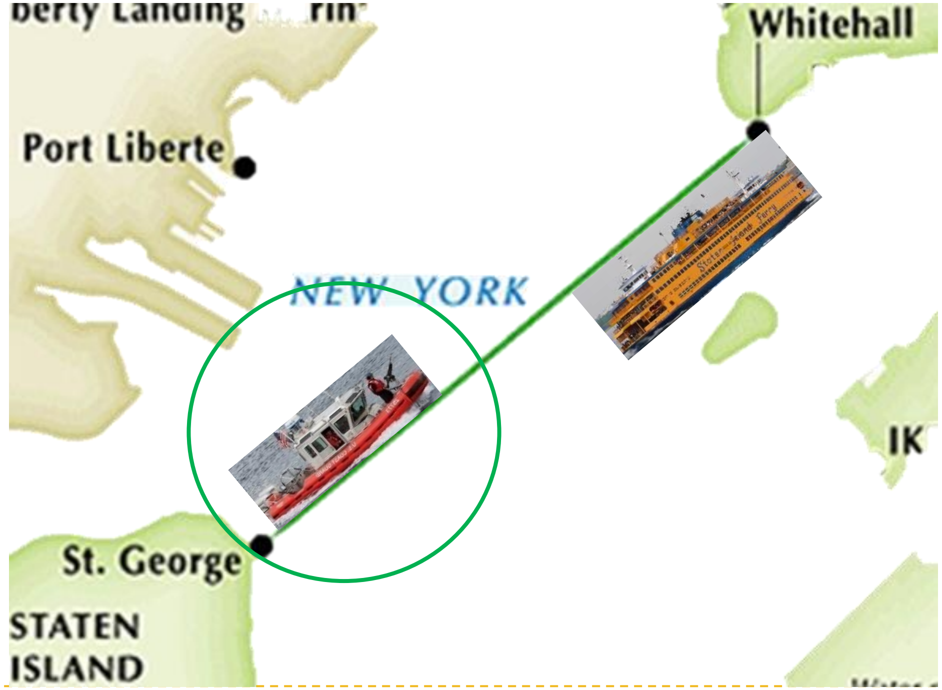
Outline

- Normal-Form Games
- Solution Concepts
- Ferry Protection

Protect Ferry Line

Problem

- Optimize the use of patrol resources
  - Moving targets: Fixed schedule
  - Potential attacks: Any time
  - Continuous time



Model

- Attacker: Which target, when to attack
- Defender: Choose a route for patrol boat
- Payoff value for attacker: $u_i(t)$ if not protected, 0 if protected
- Minimax: Minimize attacker's expected utility assume attacker best responds

Attacker's Expected Utility = Target Utility × Probability of Success

$AttEU(i,t)$ $\quad$ $u_i(t)$ $\quad$ $(1 - \sum_{r \text{ protects } t} p_r)$

Adversary

	p_r		10:00:00 AM Target 1	10:00:01 AM Target 1	…	10:30:00 AM Target 3	…
Defender	30%	Purple Route	-5, 5	-4, 4		0, 0	
	40%	Orange Route					
	20%	Blue Route					
		……					

Find Minmax Strategy

- Linear program

Probability of route

$$\min_{\boxed{p_1,p_2,\dots,p_R}} \quad v$$

$$\text{s.t. } \boxed{v \geq AttEU(i,t),}$$

Best response

For all target i, time point t

$$\sum_r p_r \leq 1$$

$$p_r \in [0,1]$$

$$AttEU(i,t) = u_i(t) \times (1 - \sum_{r\ protects\ t} p_r)$$

Challenge: Infinite routes and time points in theory!

HOW TO FIND OPTIMAL DEFENDER STRATEGY

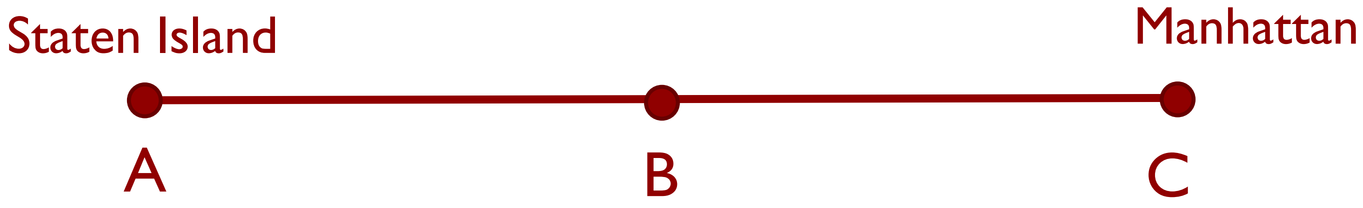
- Step 1: Compact representation for defender

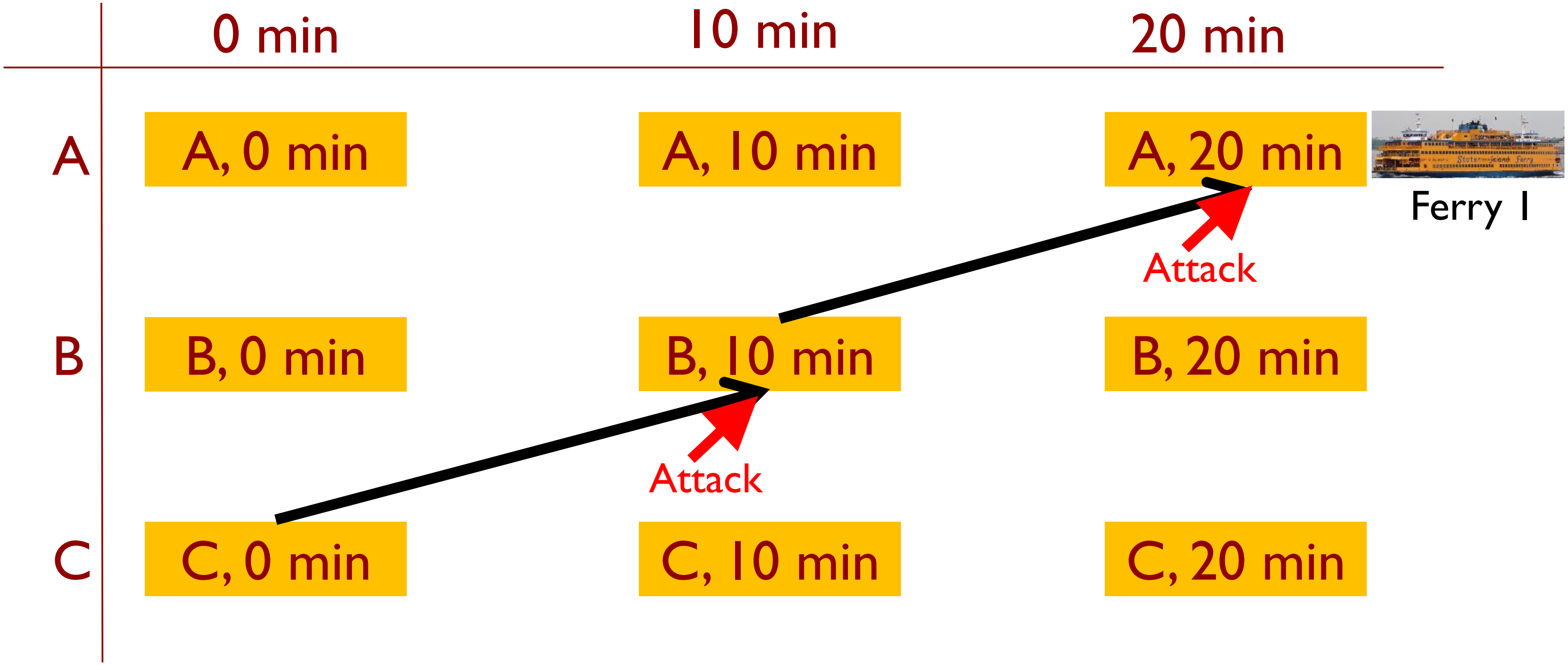
Adversary

Defender

	10:00:00 AM Target 1	**10:00:01 AM Target 1**	**…**	**10:30:00 AM Target 3**	**…**
Purple Route	-5, 5	-4, 4		0, 0	
Orange Route					
Blue Route					
……					

STEP I: COMPACT REPRESENTATION FOR DEFENDER

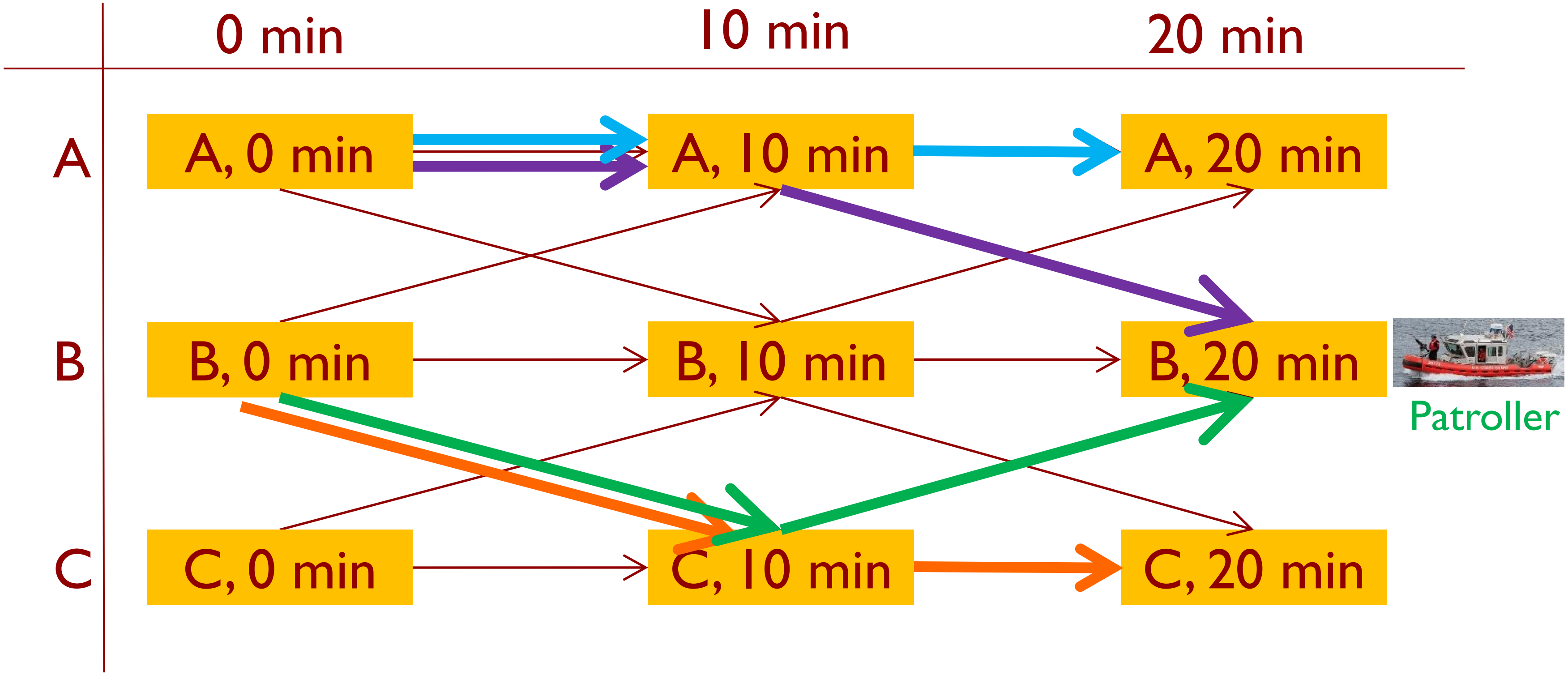




	0 min	10 min	20 min
A	A, 0 min	A, 10 min	A, 20 min
B	B, 0 min	B, 10 min	B, 20 min
C	C, 0 min	C, 10 min	C, 20 min

STEP I: COMPACT REPRESENTATION FOR DEFENDER

- Full representation: Focus on routes (N^T)
  - Prob(Orange Route) = 0.37 Prob(Green Route) = 0.33
  - Prob(Blue Route) = 0.17 Prob(Purple Route) = 0.13



STEP 1: COMPACT REPRESENTATION FOR DEFENDER

- Full representation: Focus on routes (N^T)
  - Prob(Orange Route) = 0.37 Prob(Green Route) = 0.33
  - Prob(Blue Route) = 0.17 Prob(Purple Route) = 0.13
- Linear program

$$\min_{\boxed{p_1,p_2,\ldots,p_R}} v$$

Probability of route ($O(N^T)$ variables)

s.t. $\boxed{v \geq AttEU(i,\hat{t}),}$

Best response

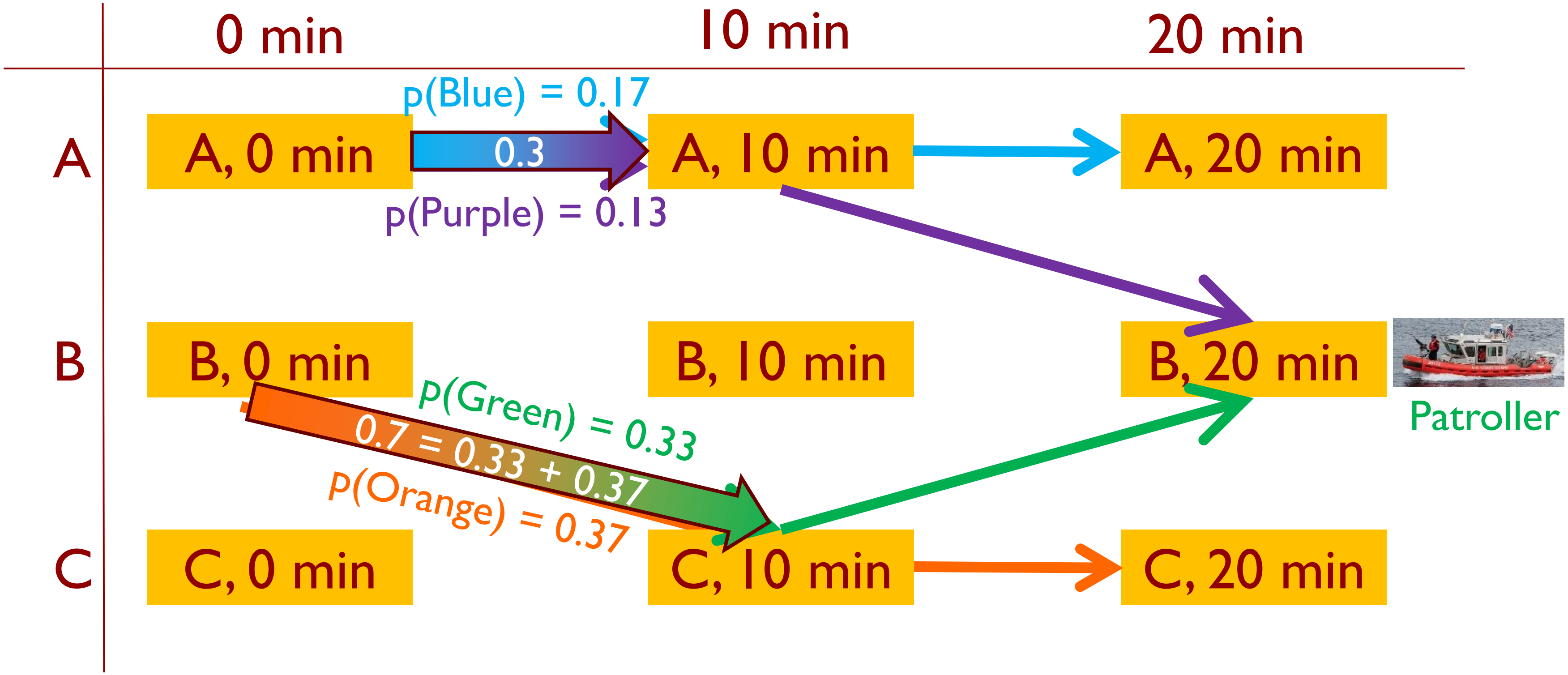
For all target i, time point $\hat{t}$

$$\sum_r p_r \leq 1$$

$$p_r \in [0,1]$$

STEP I: COMPACT REPRESENTATION FOR DEFENDER

- **Compact representation**: Focus on edges (N^2T)
  - Probability flow over each edge



STEP I: COMPACT REPRESENTATION FOR DEFENDER

Probability flow over each edge (N^2T variables)

$$\min_{f,v} v$$

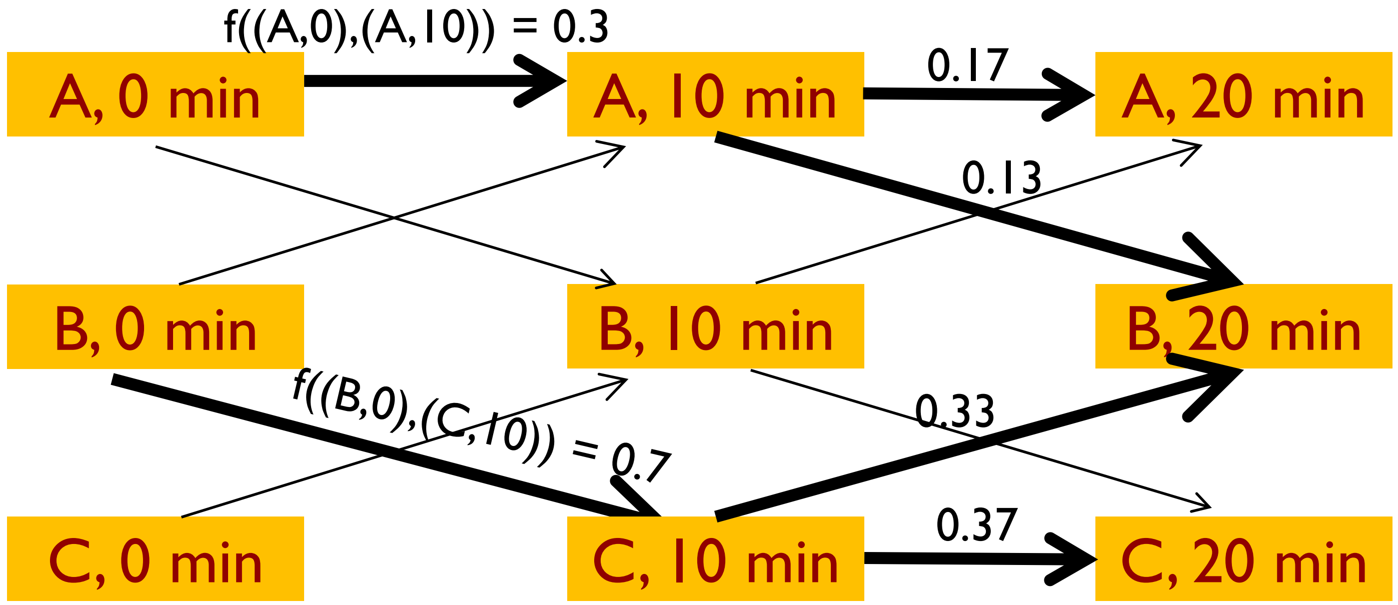
$$\text{s.t. } v \geq AttEU(i,t),$$

$$\sum_{e\in(i,t)\rightarrow} f(e) = \sum_{e\in\rightarrow(i,t)} f(e)$$

$$\sum_{e\in(*,0)\rightarrow} f(e) = 1$$

Best response

f is a unit flow



STEP I: COMPACT REPRESENTATION FOR DEFENDER

- **Theorem 1**: Let p, p' be two defender strategies in full representation, and the compact representation for both strategies is f, then

$$AttEU_p(i,t) = AttEU_{p'}(i,t)$$
$$DefEU_p(i,t) = DefEU_{p'}(i,t), \forall i, t$$

- Compact representation does not lead to any loss

Poll 2

- How many variables are needed to compute the optimal defender strategy in compact representation?
  - A: $O(N^2T)$
  - B: $O(N^T)$
  - C: $O(NT^2)$
  - D: $O(NT)$
  - E: None of the above
  - F: I don't know

HOW TO FIND OPTIMAL DEFENDER STRATEGY

- Step I: Compact representation for defender

- Step II: Compact representation for attacker

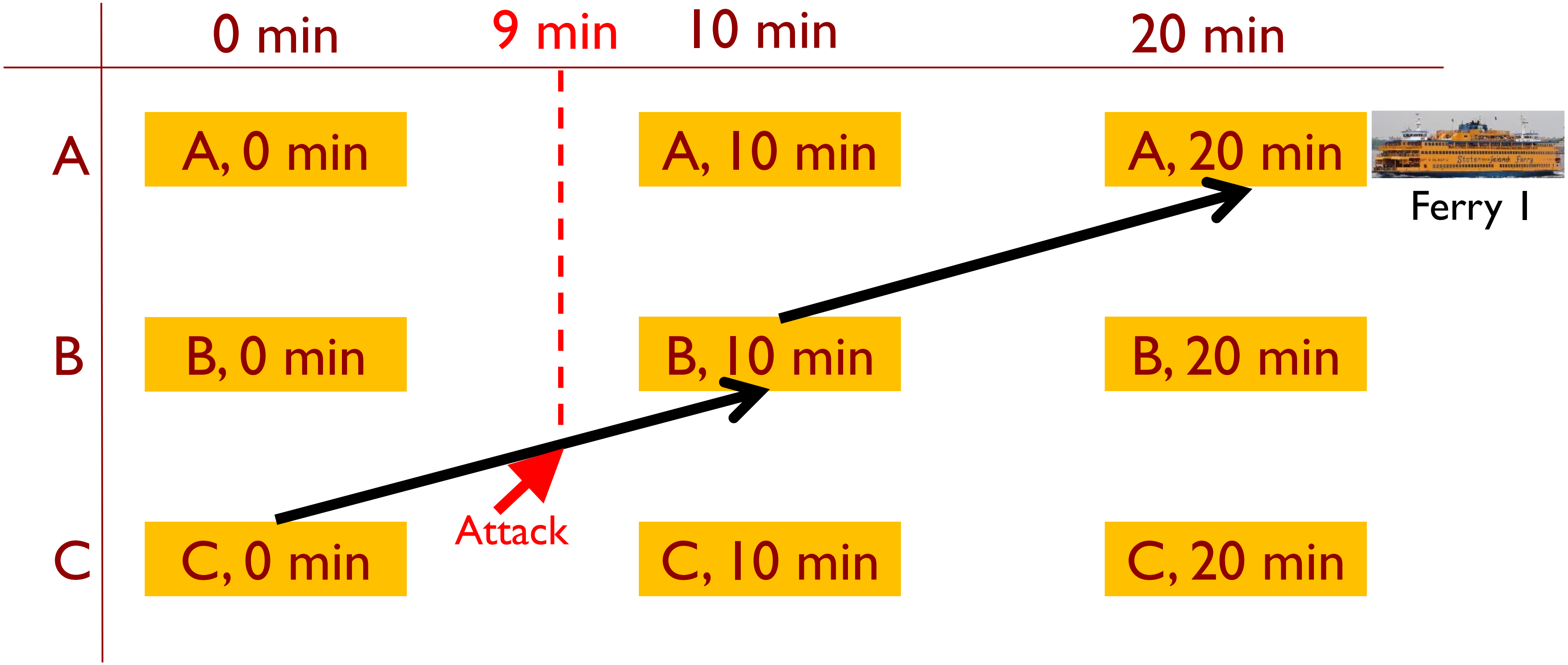
Adversary

Defender

	10:00:00 AM Target 1	10:00:01 AM Target 1	…	10:30:00 AM Target 3	…
Purple Route	-5, 5	-4, 4		0, 0	
Orange Route					
Blue Route					
……					

STEP II: COMPACT REPRESENTATION FOR ATTACKER

- **Partition attacker action set**
- Only need to reason about a few attacker actions



	0 min	9 min	10 min	20 min
A	A, 0 min		A, 10 min	A, 20 min
B	B, 0 min		B, 10 min	B, 20 min
C	C, 0 min		C, 10 min	C, 20 min

STEP II: COMPACT REPRESENTATION FOR ATTACKER

- Partition points θ^k: When protection status changes

Unprotected

Enter θ^1

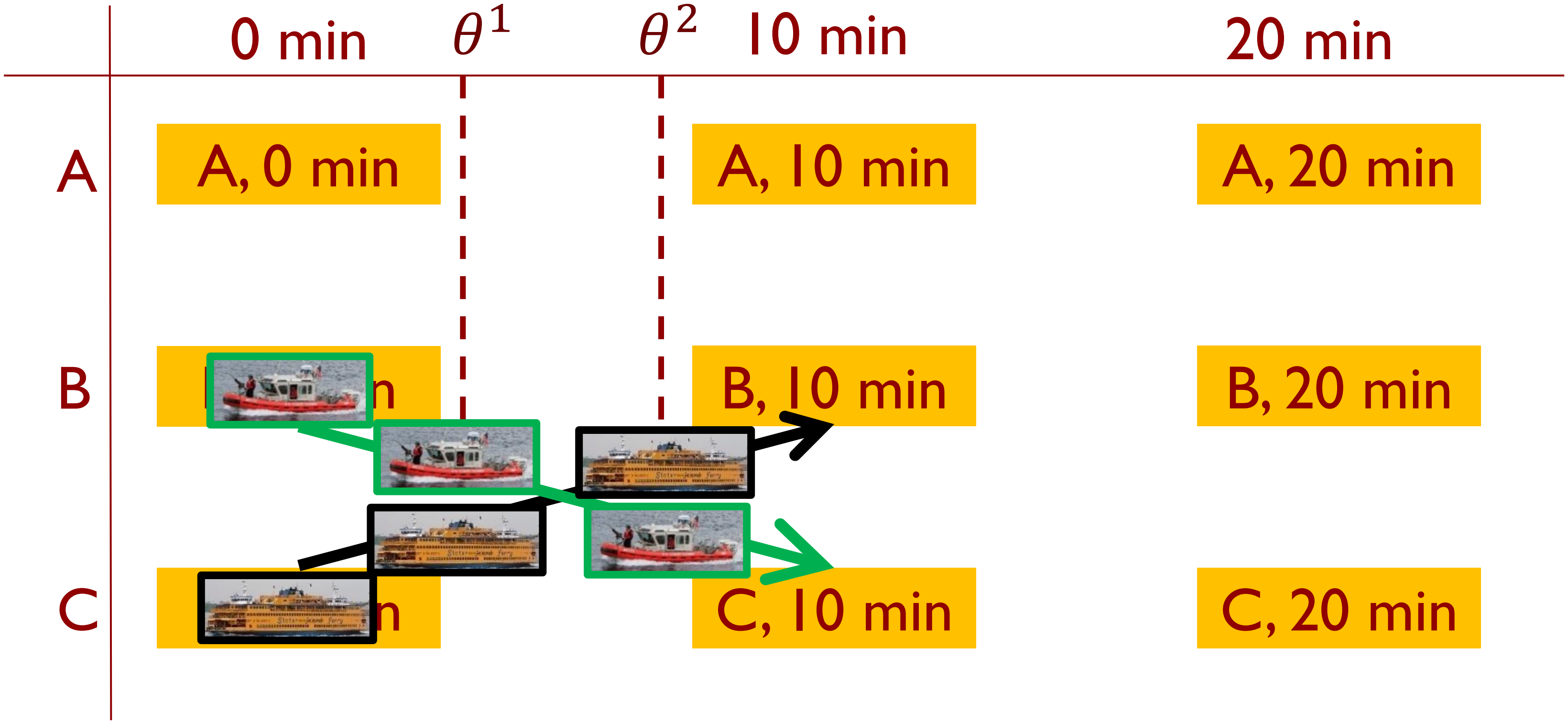
Protected

Leave θ^2

Unprotected

STEP II: COMPACT REPRESENTATION FOR ATTACKER

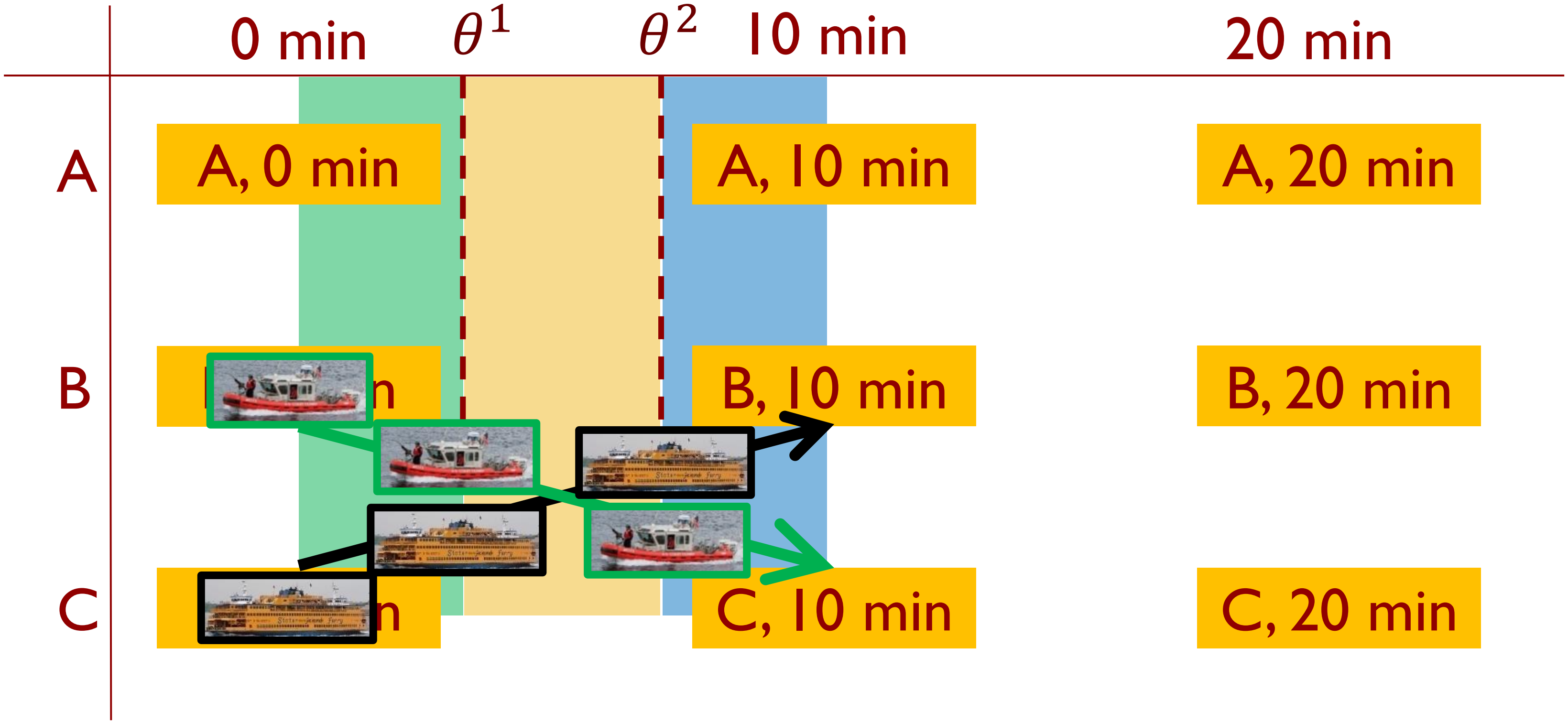
- Partition points θ^k: When protection status changes



STEP II: COMPACT REPRESENTATION FOR ATTACKER

- $AttEU = \boxed{\text{Target Utility(t)}} \times \boxed{\text{Probability of Success}}$
- **One best time point in each zone**

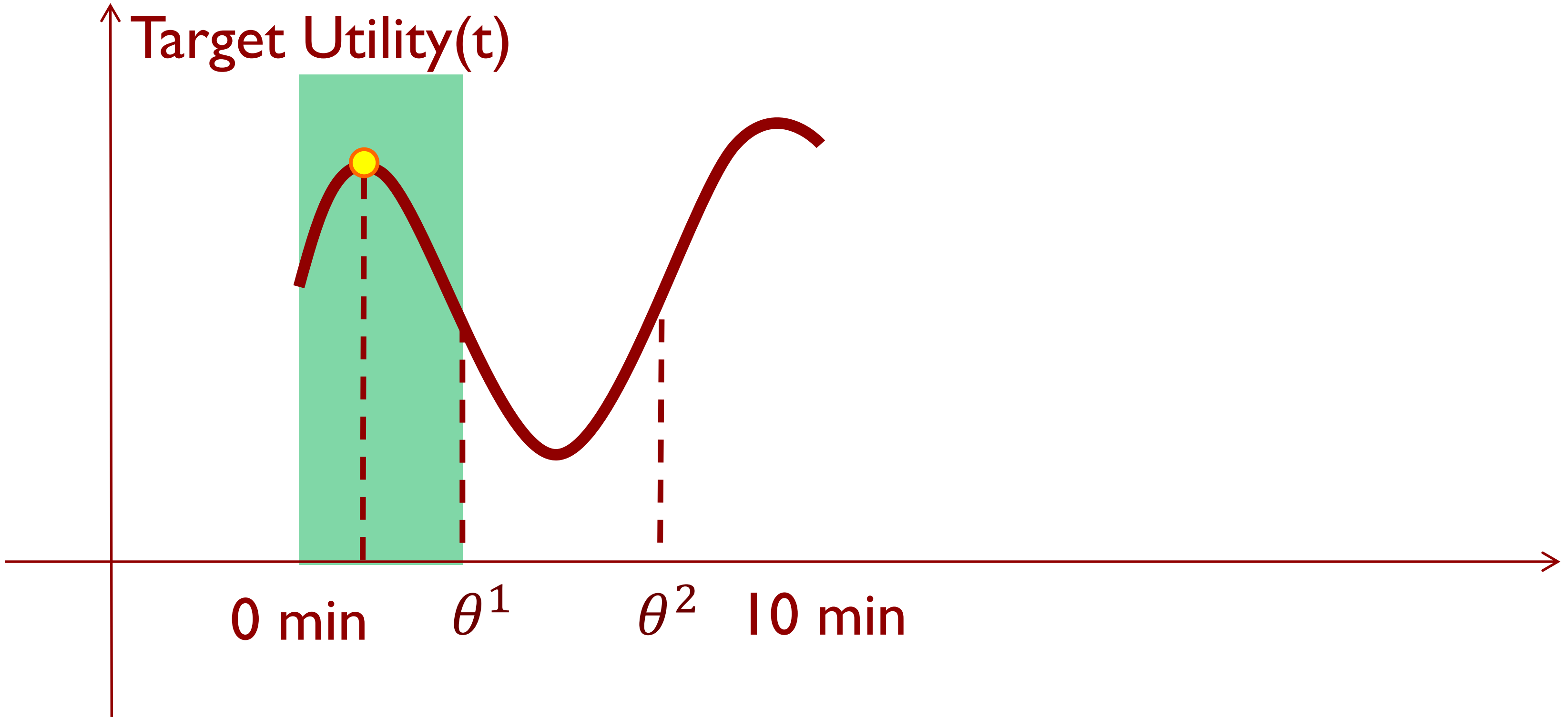
Fixed



STEP II: COMPACT REPRESENTATION FOR ATTACKER

- $AttEU = \boxed{\text{Target Utility(t)}} \times \boxed{\text{Probability of Success}}$
- <u>**One best time point in each zone**</u>

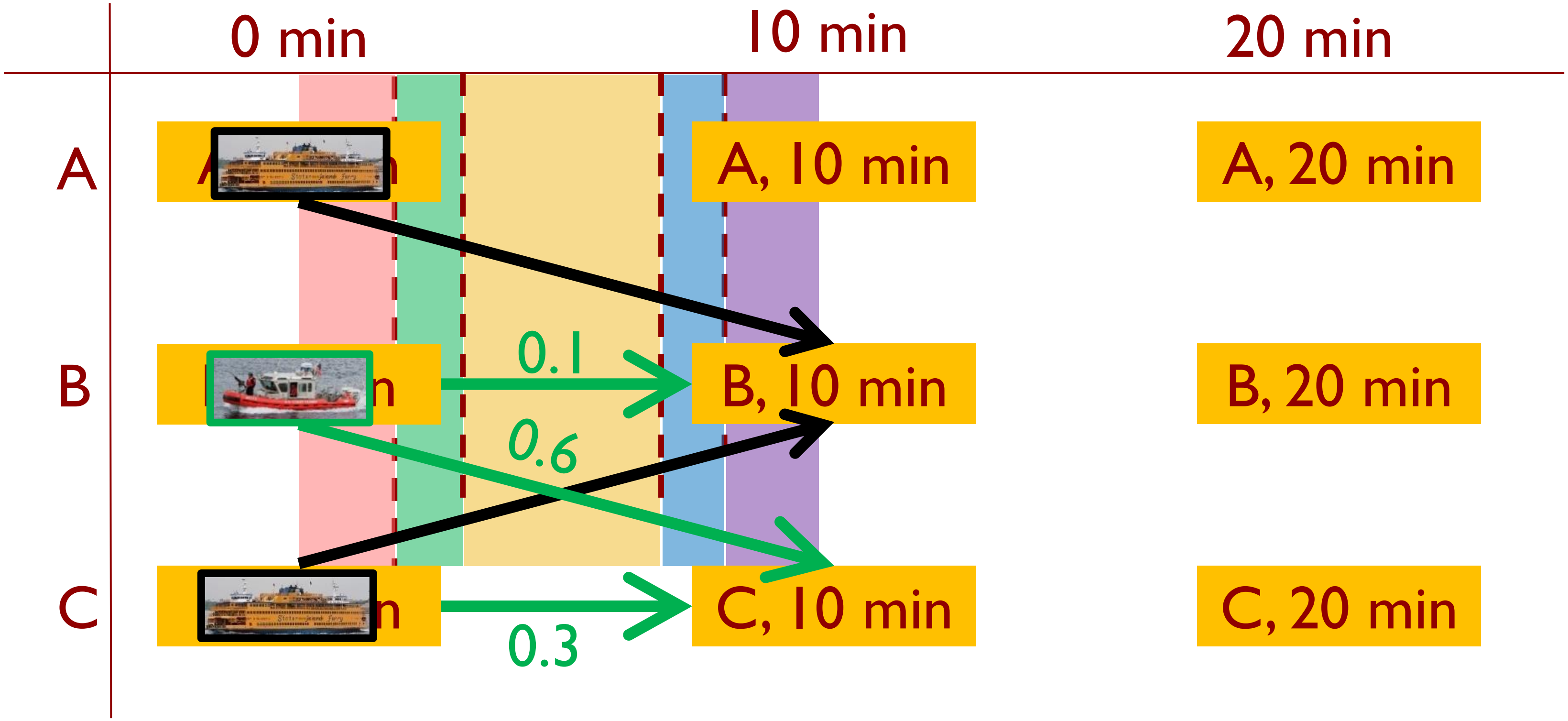
Fixed



STEP II: COMPACT REPRESENTATION FOR ATTACKER

- $AttEU = \boxed{\text{Target Utility(t)}} \times \boxed{\text{Probability of Success}}$
- <u>**One best time point in each zone**</u>

Fixed



STEP II: COMPACT REPRESENTATION FOR ATTACKER

- **Theorem 2**: Given target utility function $u_i(t)$, assume the defender's pure strategy is restricted to be a mapping from $\{\hat{\boldsymbol{t}}\}$ to $\{\widehat{\boldsymbol{d}}\}$, then in the attacker's best response, attacking time $t^* \in \{\boldsymbol{t}^*\} = \{t | \exists i, j \text{ such that } t = arg \max_{t' \in [\theta_j, \theta_{j+1}]} u_i(t')\}$

- Only considering the best time points does not lead to any loss when attacker best responds

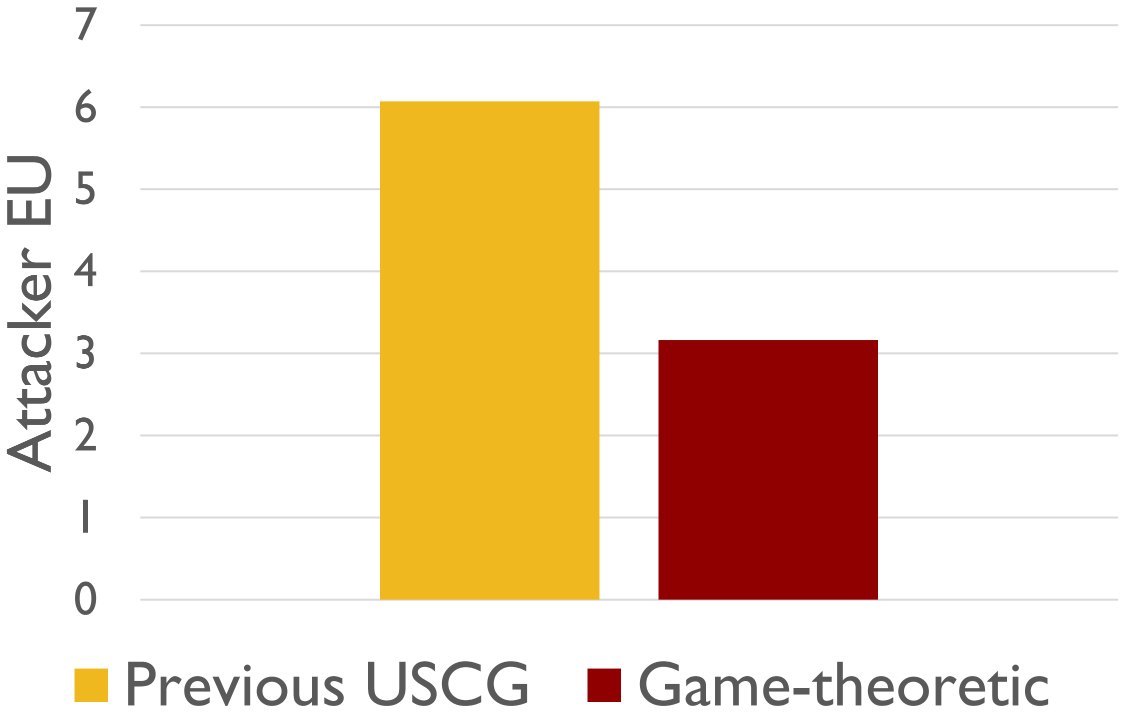
- $\infty \rightarrow O(N^2 T)$

HOW TO FIND OPTIMAL DEFENDER STRATEGY

- Step I: Compact representation for defender
- Step II: Compact representation for attacker
- Step III: Consider infinite defender action set
- Step IV: Equilibrium refinement

EVALUATION: SIMULATION RESULTS

- Randomly chosen utility function
- Attacker's expected utility (lower is better)



EVALUATION: FEEDBACK FROM REAL-WORLD

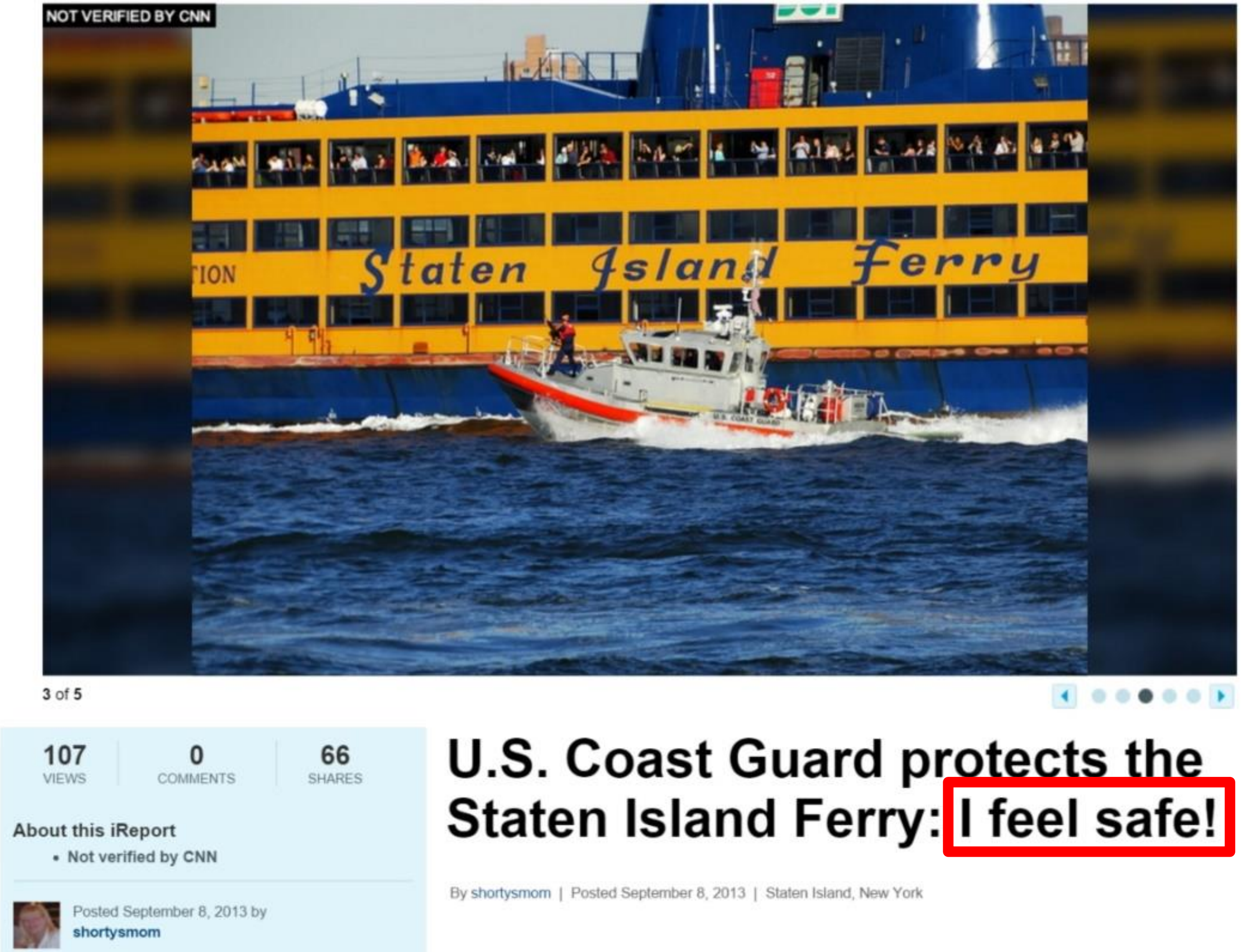
- US Coast Guard evaluation
  - Point defense to zone defense
  - Increased randomness
  - Mock attacker

- Patrollers feedback
  - More dynamic (speed change, U-turn)

- Professional mariners' observation
  - Apparent increase in Coast Guard patrols

- Used by USCG (without being forced)

PUBLIC FEEDBACK



3 of 5

107	0	66
VIEWS	COMMENTS	SHARES

About this iReport

- Not verified by CNN

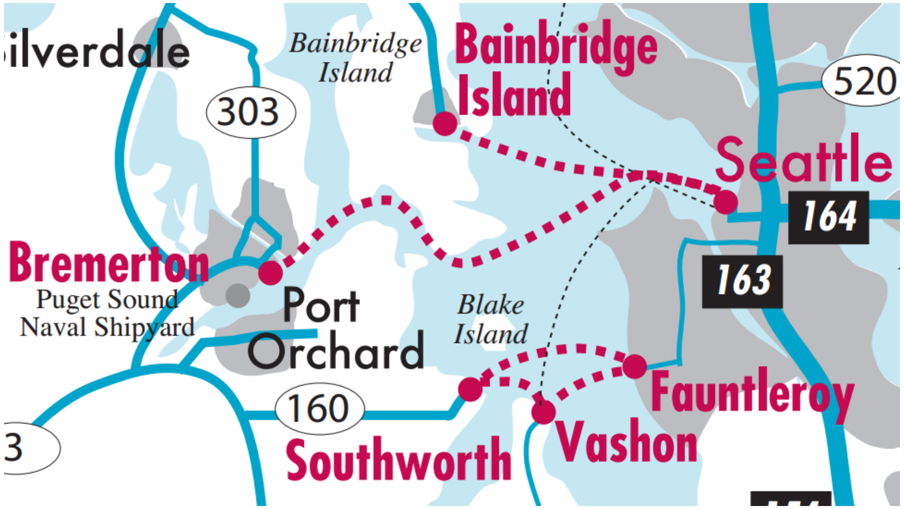
Posted September 8, 2013 by **shortysmom**

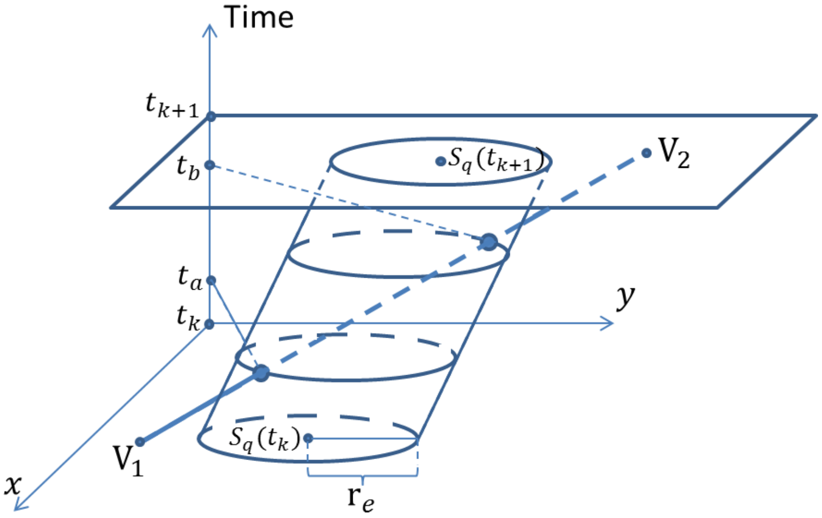
U.S. Coast Guard protects the Staten Island Ferry: I feel safe!

By shortysmom | Posted September 8, 2013 | Staten Island, New York

EXTEND TO 2-D NETWORK

- Complex ferry system: Seattle, San Francisco
- Calculate partition points in 3D space





Additional Resources and References

Additional Resources and References

- *Algorithmic Game Theory 1st Edition, Chapters 1-3*

  Noam Nisan (Editor), Tim Roughgarden (Editor), Eva Tardos (Editor), Vijay V. Vazirani (Editor)
  - http://www.cs.cmu.edu/~sandholm/cs15-892F13/algorithmic-game-theory.pdf
- Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations, Chp 3,4
- Online course
  - https://www.youtube.com/user/gametheoryonline
- *Optimal Patrol Strategy for Protecting Moving Targets with Multiple Mobile Resources*

Backup Slides

Minimax Strategy

- Minimax Strategy in n-player games:
  - Coordinate with other players to minimize best case expected utility for a particular player (just want to harm that player)
  - Minimax strategy for player i against player j is i's component of s_{-j} in $\operatorname{argmin}_{s_{-j}} \max_{s_j} u_j(s_j, s_{-j})$
  - Minimax value for player j is $\min_{s_{-j}} \max_{s_j} u_j(s_j, s_{-j})$
  - Focus on single player's strategy
  - Can be computed through linear programming (treating all players other than j as a meta-player)

Find All NEs

- Recall: A mixed strategy is BR iff all actions in the support are BR
- To find all NEs, think from the inverse direction: enumerate support
  - If we know in a NE, for player i, action $1, 2$, and 3 are in the support of s_i, action $4, 5$ are not what does it mean?
    - (1)
    - (2)
    - (3)
    - (4)

Find All NEs

- Recall: A mixed strategy is BR iff all actions in the support are BR
- To find all NEs, think from the inverse direction: enumerate support
  - If we know in a NE, for player i, action $1, 2,$ and 3 are in the support of s_i, action $4, 5$ are not what does it mean?
    - (1) Action $1, 2,$ and 3 are chosen with non-zero probability, action 4,5 are chosen with zero probability
    - (2) The probability of choosing action $1, 2, 3$ sum up to 1
    - (3) Action $1, 2,$ and 3 lead to the exactly same expected utility
    - (4) The expected utility of taking action $1, 2,$ and 3 is not lower than action $4, 5$

Find All NEs

- If support for both Alex and Berry is (F, C), then action F and C should lead to same expected utility for Alex when fixing Berry’s strategy and vice versa
- Assume Alex’s strategy is $s_A = (x_1, x_2)$ and Berry’s strategy is $s_B = (y_1, y_2)$ then similar to (1)-(4) in the previous slide, we know

		Football	Concert
Alex	Football	2,1	0,0
	Concert	0,0	1,2

Find All NEs

- If support for both Alex and Berry is (F, C), then action F and C should lead to same expected utility for Alex when fixing Berry's strategy and vice versa
- Assume Alex's strategy is $s_A = (x_1, x_2)$ and Berry's strategy is $s_B = (y_1, y_2)$ then similar to (1)-(4) in the previous slide, we know

(1): $x_1 > 0, x_2 > 0, y_1 > 0, y_2 > 0$
(2): $x_1 + x_2 = 1, y_1 + y_2 = 1$
(3): $u_A(F, s_B) = u_A(C, s_B), u_B(s_A, F) = u_B(s_A, C)$

$$u_A(F, s_B) = 2 \times y_1 + 0 \times y_2$$
$$u_A(C, s_B) = 0 \times y_1 + 1 \times y_2$$

So $2y_1 = y_2$

$$u_B(s_A, F) = 1 \times x_1 + 0 \times x_2$$
$$u_B(s_A, C) = 0 \times x_1 + 2 \times x_2$$

So $x_1 = 2x_2$

Alex

	Football	Concert
Football	2,1	0,0
Concert	0,0	1,2

Solve the equations in (2)(3) and get $s_A = \left(\frac{2}{3}, \frac{1}{3}\right), s_B = \left(\frac{1}{3}, \frac{2}{3}\right)$ which satisfy (1). It is indeed a NE with specified support.

Find All NEs

- Support Enumeration Method (for bimatrix games)
  - Enumerate all support pairs with the same size for size=1 to $\min_{i} |A_i|$
  - For each possible support pair J_1, J_2, build and solve a LP
    - An NE is found if the LP has a feasible solution

Find All NEs

- Support Enumeration Method (for bimatrix games)
  - Enumerate all support pairs with the same size for size=1 to $\min_{\mathrm{i}} |A_i|$
  - For each possible support pair J_1, J_2, build and solve a LP

$$
\begin{aligned}
&\max_{x,y,v} 1\\
&x_i \geq 0, \forall i; y_j \geq 0, \forall j\\
&x_i = 0, \forall i \notin J_1; y_j = 0, \forall j \notin J_2\\
&\textstyle\sum_{i \in J_1} x_i = 1\\
&\textstyle\sum_{j \in J_2} y_j = 1\\
&\textstyle\sum_{j \in J_2} y_j u_1(i,j) = v_1, \forall i \in J_1\\
&\textstyle\sum_{i \in J_1} x_i u_2(i,j) = v_2, \forall j \in J_2\\
&\textstyle\sum_{j \in J_2} y_j u_1(i,j) \leq v_1, \forall i \notin J_1\\
&\textstyle\sum_{i \in J_1} x_i u_2(i,j) \leq v_2, \forall j \notin J_2
\end{aligned}
$$

    - An NE is found if the LP has a feasible solution

Find All NEs

- Support Enumeration Method (for bimatrix games)
  - Enumerate all support pairs with the same size for size=1 to $\min_{\mathrm{i}} |A_i|$
  - For each possible support pair J_1, J_2, build and solve a LP
    - Variables: $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n, v_1, v_2$
    - Objective: a dummy one $\max_{x,y,v} 1$
    - Constraints (1b,1c): Probabilities are nonnegative, probability of actions not in the support is zero
      - ☐ $x_i \geq 0, \forall i; y_j \geq 0, \forall j; x_i = 0, \forall i \notin J_1; y_j = 0, \forall j \notin J_2$
    - Constraints (2): Probability of taking actions in the support sum up to 1
      - ☐ $\sum_{i \in J_1} x_i = 1; \sum_{j \in J_2} y_j = 1$
    - Constraints (3): Expected utility (EU) of choosing any action is the support is the same when fixing the other player's strategy
      - ☐ $\sum_{j \in J_2} y_j u_1(i,j) = v_1, \forall i \in J_1; \sum_{i \in J_1} x_i u_2(i,j) = v_2, \forall j \in J_2$
    - Constraints (4): Actions not in support does not lead to higher expected utility
      - ☐ $\sum_{j \in J_2} y_j u_1(i,j) \leq v_1, \forall i \notin J_1; \sum_{i \in J_1} x_i u_2(i,j) \leq v_2, \forall j \notin J_2$
    - An NE is found if the LP has a feasible solution

Compute Nash Equilibrium

- Find all Nash Equilibrium (two-player)
  - Support Enumeration Method
  - Lemke-Howson Algorithm
    - Linear Complementarity (LCP) formulation (another special class of optimization problem)
    - Solve by pivoting on support (similar to Simplex algorithm)
  - In practice, available solvers/packages: nashpy (python), gambit project (http://www.gambit-project.org/)